Application of Derivatives

EXERCISES

ELEMENTRY

Q.1 (1)

Velocity, $v^2 = 2 - 3x$

Differentiating with respect to *t*, we get

$$2v\frac{dv}{dt} = -3.\frac{dx}{dt} \Rightarrow 2v\frac{dv}{dt} = -3v \Rightarrow \frac{dv}{dt} = -\frac{3}{2}$$

Hence acceleration is uniform.

Q.2 (1)

Displacements $s = -4t^2 + 2t$

Now velocity $v = -8t + 2$ and its acceleration $a = -8$

So $\left(\frac{ds}{dt}\right)_{t=1/2} = -8 \times \frac{1}{2} + 2 = -2$ and

Q.3 (1)

Given that $dV/dt = 30ft^3/min$ and $r = 15ft$

$$V = \frac{4}{3}\pi r^3; \ \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} = \frac{30}{4 \times \pi \times 15 \times 15} = \frac{1}{30\pi}\ ft/min$$

Q.4 (3)

Given $y^2 = 2(x-3)$ .....(i)

Differentiate *w.r.t. x*, $2y.\frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx} = \frac{1}{y}$

Slope of the normal $= \frac{-1}{\left(\frac{dy}{dx}\right)} = -y$

Slope of the given line = 2

$\therefore y = -2$

From equation (i), x = 5

$\therefore$ Required point is (5 –2).

Q.5 (1)

Given curve $x^2 = 3 - 2y$ ...(i)

Differentiate *w.r.t. x*, $2x = 0 - 2\frac{dy}{dx}\ \frac{dy}{dx} = -x$

Slope of the tangent of the curve $= -x$

From the given line, slope $= -1$, $\therefore x = 1$ and from equation (i), $y = 1$. $\therefore$ Co-ordinate of the point is (1, 1).

Q.6 (3)

$y = 2\cos x$

At $x = \frac{\pi}{4}$, $y = \frac{2}{\sqrt{2}} = \sqrt{2}$ and $\frac{dy}{dx} = -2.\sin x$

$$\therefore \left(\frac{dy}{dx}\right)_{x=\pi/4} = -\sqrt{2}$$

$\therefore$ Equation of tangent at $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is

$$y - \sqrt{2} = -\sqrt{2}\left(x - \frac{\pi}{4}\right)$$

Q.7 (4)

$$y = x\log x \Rightarrow \frac{dy}{dx} = 1 + \log x$$

The slope of the normal $= -\frac{1}{(dy/dx)} = \frac{-1}{1+\log x}$

The slope of the line $2x - 2y = 3$ is 1.

$$\therefore \frac{-1}{1+\log x} = 1 \Rightarrow \log x = -2 \Rightarrow x = e^{-2}$$

$\therefore\ y = -2e^{-2}$

$\therefore$ Co-ordinate of the point is $(e^{-2}, -2e^{-2})$.

Q.8 (3)

Slope of the normal $= -\frac{1}{\left(\frac{dy}{dx}\right)}$

This is parallel to *x*-axis $\Rightarrow -\frac{1}{\left(\frac{dy}{dx}\right)} = 0 \Rightarrow \frac{dx}{dy} = 0$

Q.9 (1)

Let the point of contact be (h, k),

where $k = h^4$. Tangent is $y - k = 4h^3(x - h)$,

$$\left[\because \frac{dy}{dx} = 4x^3\right]$$

It passes through (2, 0), $\therefore\ -k = 4h^3(2-h)$

$\Rightarrow h = 0$ or 8/3, $\therefore\ k = 0$ or $(8/3)^4$

$\therefore$ Points of contact are (0, 0) and $\left(\frac{8}{3},\left(\frac{8}{3}\right)^4\right)$

$\therefore$ Equation of tangents are

$y = 0$ and $y-\left(\frac{8}{3}\right)^4 = 4\left(\frac{8}{3}\right)^3\left(x-\frac{8}{3}\right)$.

Q.10 (4)

$x^2 = -4y \qquad \Rightarrow 2x = -4\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{2} \qquad \Rightarrow \left(\frac{dy}{dx}\right)_{(-4,-4)} = 2$.

We know that equation of tangent is,

$(y-y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x-x_1) \Rightarrow y+4 = 2(x+4)$

$\Rightarrow 2x-y+4=0$.

Q.11 (3)

$x^3-8a^2y=0 \Rightarrow 3x^2-8a^2.\frac{dy}{dx}=0$

$\Rightarrow 3x^2 = 8a^2.\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{3x^2}{8a^2}$

$\therefore$ Slope of the normal $= -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{1}{\frac{3x^2}{8a^2}} = -\frac{8a^2}{3x^2}$

Given $\frac{-8a^2}{3x^2} = \frac{-2}{3} \quad \therefore (x, y) = (2a, a)$.

Q.12 (3)

Length of normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Now,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$\left(\frac{dy}{dx}\right)_{\left(\theta=\frac{\pi}{2}\right)} = \left[\tan\frac{\theta}{2}\right]_{\left(\theta=\frac{\pi}{2}\right)} = 1[y]_{\left(\theta=\frac{\pi}{2}\right)} = a\left(1-\cos\frac{\pi}{2}\right) = a$$

$\therefore$ Length of normal $= a\sqrt{1+(1)^2} = \sqrt{2}a$

Q.13 (3) $x = a(t+\sin t)$, $y = a(1-\cos t)$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a(\sin t)}{a(1+\cos t)} = \tan\frac{t}{2}$

Length of the normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$= a(1-\cos t)\sqrt{1+\tan^2(t/2)} = a(1-\cos t)\sec(t/2)$

$= 2a\sin^2(t/2)\sec(t/2) = 2a\sin(t/2)\tan(t/2)$.

Q.14 (2)

$by^2 = (x+a)^3 \Rightarrow 2by.\frac{dy}{dx} = 3(x+a)^2 \Rightarrow \frac{dy}{dx} = \frac{3}{2by}(x+a)^2$

$\therefore$ Subnormal $= y\frac{dy}{dx} = \frac{3}{2b}(x+a)^2$

$\therefore$ Subtangent $= \frac{y}{\left(\frac{dy}{dx}\right)} = \frac{y}{\frac{3(x+a)^2}{2by}} = \frac{2by^2}{3(x+a)^2}$

$= \frac{2b\frac{(x+a)^3}{b}}{3(x+a)^2} = \frac{2}{3}(x+a)$

$\therefore$ (Subtangent)$^2 = \frac{4}{9}(x+a)^2$

and $\frac{(\text{Subtangent})^2}{\text{Subnormal}} = \frac{\frac{4}{9}(x+a)^2}{\frac{3}{2b}(x+a)^2} = \frac{8b}{27}$

$\Rightarrow$ (Subtangent)^2 = constant $\times$ (Subnormal).

$\therefore$ (Subtangent)$^2 \propto$ (Subnormal).

Q.15 (1)

$\frac{dx}{d\theta} = a(1+\cos\theta), \frac{dy}{d\theta} = a(\sin\theta)$

$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = 1, \quad y|_{\theta=\frac{\pi}{2}} = a$

Length of sub-tangent $ST = \frac{y}{dy/dx} = \frac{a}{1} = a$.

and length of sub-normal $SN = y\frac{dy}{dx} = a.1 = a$

Hence $ST = SN$.

Q.16 (1)

For curve $y^2 = 4x \Rightarrow \frac{dy}{dx} = \frac{4}{2y}$

$\therefore \left(\frac{dy}{dx}\right)_{(1,2)} = 1$ and for curve

$x^2 + y^2 = 5 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

$\therefore \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{-1}{2}$

$\therefore$ Angle between the curves is

$$\theta = \tan^{-1}\left|\frac{\frac{-1}{2}-1}{1+\left(\frac{-1}{2}\right)}\right| = \tan^{-1}(3)$$

Q.17 (4)

$y = x^2 \Rightarrow \frac{dy}{dx} = m_1 = 2x$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = 2 = m_1$ and $x = y^2 \Rightarrow 1 = 2y\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = m_2 = \frac{1}{2y} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{1}{2}$

$\therefore$ Angle of intersection, $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$$= \frac{2 - \frac{1}{2}}{1 + 2 \times \frac{1}{2}} = \frac{3}{4} \Rightarrow \theta = \tan^{-1}(3/4)$$

Q.18 (3)

$y = x^2 \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = m_1 = 2x = 2$

$6y = 7 - x^3 \Rightarrow 6.\frac{dy}{dx} = -3x^2$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = m_2 = -\frac{1}{2}$

Clearly $m_1 m_2 = -1$, therefore angle of intersection is $\frac{\pi}{2}$.

Q.19 (2)

Let $y = f(x) = x^2 e^{-x}$

$\Rightarrow \frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x} = e^{-x}(2x - x^2)$

Hence $f'(x) \geq 0$ for every $x \in [0,2]$ therefore it is non-decreasing in [0,2].

Q.20 (4)

$f(x) = -2x^3 - 9x^2 - 12x + 1$

$\Rightarrow f'(x) = -6x^2 - 18x - 12$

To be decreasing $f'(x) < 0$, *i.e.*, $-6x^2 - 18x - 12 < 0$

$\Rightarrow x^2 + 3x + 2 > 0 \Rightarrow (x+2)(x+1) > 0$

Therefore either $x < -2$ or $x > -1$

$\Rightarrow x \in (-1, \infty)$ or $(-\infty, -2)$

Q.21 (1)

$f(x) = x^4 - \frac{x^3}{3} \Rightarrow f'(x) = 4x^3 - x^2$

For increasing $4x^3 - x^2 > 0 = x^2(4x - 1) > 0$

Therefore, the function is increasing for $x > \frac{1}{4}$

Similarly decreasing for $x < \frac{1}{4}$.

Q.22 (1)

$f(x) = 5^{-x}$

$\Rightarrow f'(x) = -5^{-x}\log_e 5 = -\frac{\log_e 5}{5^x}$

$\Rightarrow f'(x) < 0$ for all x

i.e., f(x) is decreasing for all x.

Q.23 (3)

$f(x) = \sin x - \frac{x}{2} \Rightarrow f'(x) = \cos x - \frac{1}{2}$

$f'(x) > 0$ for increasing function

Obviously it is increasing for $-\frac{\pi}{3} < x < \frac{\pi}{3}$.

Q.24 (2)

Given $f(x) = x^3 - x^2 - x - 4$

This function will be decreasing function when $f'(x) < 0$

$\Rightarrow 3x^2 - 2x - 1 < 0 \Rightarrow 3x^2 - 3x + x - 1 < 0$

$\Rightarrow (3x+1)(x-1)<0$; $\therefore 3x+1>0$ and $x-1<0$

$x>-\frac{1}{3}$ and $x<1$: $\therefore x \in \left(\frac{-1}{3},1\right)$

Q.25 (4) $f(x)=(x+2)e^{-x}$

$f'(x)=e^{-x}-e^{-x}(x+2)$

$f'(x)=-e^{-x}[x+1]$

For increasing, $-e^{-x}(x+1)>0$ or $e^{-x}(x+1)<0$

$e^{-x}>0$ $(x+1)<0$

$x\in(-\infty,\infty)$ and $x\in(-\infty,-1)$

$\therefore x\in(-\infty,-1)$

Hence, the function is increasing in $(-\infty,-1)$

For decreasing, $-e^{-x}(x+1)<0$ or $e^{-x}(x+1)>0$,

$x\in(-1,\infty)$

Hence the function is decreasing in $(-1,\infty)$.

Q.26 (3)

$f(x)=x^3-10x^2+200x-10$

$f'(x)=3x^2-20x+200$

For increasing $f'(x)>0 \Rightarrow 3x^2-20x+200>0$

$$3\left[x^2-\frac{20}{3}x+\frac{200}{3}+\frac{100}{9}-\frac{100}{9}\right]>0$$

$$\Rightarrow 3\left[\left(x-\frac{10}{3}\right)^2+\frac{500}{9}\right]>0$$

$$\Rightarrow 3\left(x-\frac{10}{3}\right)^2+\frac{500}{3}>0$$

Always increasing throughout real line.

Q.27 (4)

The function is monotonic increasing, if $f'(x)>0$

$$\Rightarrow \frac{(2\sin x+3\cos x)(\lambda\cos x-6\sin x)}{(2\sin x+3\cos x)^2}$$

$$-\frac{(\lambda\sin x+6\cos x)(2\cos x-3\sin x)}{(2\sin x+3\cos x)^2}>0$$

$\Rightarrow 3\lambda(\sin^2 x+\cos^2 x)-12(\sin^2 x+\cos^2 x)>0$

$\Rightarrow 3\lambda-12>0 \Rightarrow \lambda>4.$

Q.28 (4) $f(x)=\sin x-\cos x$

$$f'(x)=\cos x+\sin x=\sqrt{2}\left[\cos\left(x-\frac{\pi}{4}\right)\right]=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$$

For $f(x)$ decreasing, $f'(x)<0$

$\frac{\pi}{2}<\left(x-\frac{\pi}{4}\right)<\frac{3\pi}{2}$, (within $0\le x\le 2\pi$).

$\Rightarrow \frac{3\pi}{4}<x\le\frac{7\pi}{4}$.

Q.29 (2) $f(x)=\frac{\log x}{x}$

$$f'(x)=\frac{1}{x^2}-\frac{\log x}{x^2}=\frac{1-\log x}{x^2}$$

For $f(x)$ to be increasing, $f'(x)>0$

$\Rightarrow 1-\log x>0 \Rightarrow 1>\log x \Rightarrow e>x$

$\therefore f(x)$ is increasing in the interval $(0,e)$.

Q.30 (2)

$$f(x)=\frac{1}{x+1}-\log(1+x) \quad \Rightarrow$$

$$f'(x)=-\frac{1}{(x+1)^2}-\frac{1}{1+x}$$

$$f'(x)=-\left[\frac{1}{x+1}+\frac{1}{(x+1)^2}\right]$$

$f'(x)=-ve$, when $x>0$ or $f'(x)<0$, $\forall x>0$

$\therefore f(x)$ is decreasing function.

Q.31 (4)

Let $f(x)=2x^3-24x+107$

At $x=-3, f(-3)=2(-3)^3-24(-3)+107=125$

At $x=3$, $f(3)=2(3)^3-24(3)+107=89$

For maxima or minima, $f'(x)=6x^2-24=0$

$\Rightarrow x=2, -2$

So at $x=2, f(2)=2(2)^3-24(2)+107=75$

at $x=-2$, $f(-2)=2(-2)^3-24(-2)+107=139$

Thus the maximum value of the given function in [–3, 3] is 139.

Q.32 (3)

$f(x)=2x^2+x-1$

$\Rightarrow f'(x)=4x+1 \Rightarrow f'(x)=0 \Rightarrow x=-\frac{1}{4}$

$f''(x)=4=+ve$

$$\therefore [f(-1/4)]_{min}=\frac{2}{16}-\frac{1}{4}-1=\frac{-9}{8}.$$

Q.33 (2)

$f(x)=\cos x+\cos(\sqrt{2}x)$

$f'(x)=-\sin x-\sqrt{2}\sin(\sqrt{2}x)=0$

Hence $x=0$ is the only solution.

$f''(x)=-\cos x-2\cos(\sqrt{2}x)<0$ at $x=0$

Hence maxima occurs at $x=0$.

Q.34 (1)

Let $y=x^x \Rightarrow \frac{dy}{dx}=x^x(1+\log x)$

For $\frac{dy}{dx}>0$; $x^x(1+\log x)>0 \Rightarrow$

$1+\log x>0 \Rightarrow \log_e x>\log_e \frac{1}{e}$

For this to be positive, *x* should be greater than $\frac{1}{e}$.

Q.35 (4)

Here $f(x)=\frac{x^2-3x}{x-1} \Rightarrow f'(x)=\frac{x^2-2x+3}{(x-1)^2}$

Obviously, it is not derivable at $x=1$ *i.e.*, in $(0,3)$

Also $f(a)=f(b)$ does not hold for $[-3,0]$ and [1. 5, 3] Hence the answer is (4).

Q.36 (1)

$f(1)=f(3)\Rightarrow a+b-5=3a+b-27\Rightarrow a=11$ which is given in option (1) only.

Q.37 (1) $f(x)=e^{-2x}\sin 2x$

$\Rightarrow f'(x)=2e^{-2x}(\cos 2x-\sin 2x)$

Now, $f'(c)=0$

$\Rightarrow \cos 2c-\sin 2c=0 \Rightarrow \tan 2c=1 \Rightarrow c=\frac{\pi}{8}$

Q.38 (3)

We know that $f'(c)=\frac{f(b)-f(a)}{b-a}$

$\Rightarrow f'(c)=\frac{0-1}{\pi/2}=-\frac{2}{\pi}$ .....(i)

But $f'(x)=-\sin x \Rightarrow f'(c)=-\sin c$ ....(ii)

From (i) and (ii), we get

$-\sin c=-\frac{2}{\pi}\Rightarrow c=\sin^{-1}\left(\frac{2}{\pi}\right)$.

Q.39 (3)

From mean value theorem $f'(c)=\frac{f(b)-f(a)}{b-a}$

$a=0, f(a)=0 \Rightarrow b=\frac{1}{2}, f(b)=\frac{3}{8}$

$f'(x)=(x-1)(x-2)+x(x-2)+x(x-1)$

$f'(c)=(c-1)(c-2)+c(c-2)+c(c-1)$

$= c^2-3c+2+c^2-2c+c^2-c$

$f'(c)=3c^2-6c+2$

According to mean value theorem,

$f'(c)=\frac{f(b)-f(a)}{b-a}$

$\Rightarrow 3c^2-6c+2=\frac{(3/8)-0}{(1/2)-0}=\frac{3}{4}$

$\Rightarrow 3c^2-6c+\frac{5}{4}=0$

$c=\frac{6\pm\sqrt{36-15}}{2\times 3}=\frac{6\pm\sqrt{21}}{6}=1\pm\frac{\sqrt{21}}{6}$

Q.40 (2)

$f(b)=f(2)=8-24a+10=18-24a$

$f(a)=f(1)=1-6a+5=6-6a$

$f'(x)=3x^2-12ax+5$

From Lagrange's mean value theorem,

$f'(x)=\frac{f(b)-f(a)}{b-a}=\frac{18-24a-6+6a}{2-1}$

$\therefore f'(x)=12-18a$

At $x=\frac{7}{4}, 3\times\frac{49}{16}-12a\times\frac{7}{4}+5=12-18a$

$\Rightarrow 3a=\frac{147}{16}-7 \Rightarrow 3a=\frac{35}{16} \Rightarrow a=\frac{35}{48}$

Q.41 (4)

Let $f(x)=x^2\log x \Rightarrow f'(x)=2x\log x+x$

and $f''(x)=2(1+\log x)+1$

Now $f''(1)=3+2\log_e 1$ and $f''(e)=3+2\log_e e$

$f(x)$ has local minimum at $\frac{1}{\sqrt{e}}$, but x lies only in interval $(1,e)$ so that has not extremum in

Hence neither a point of maximum nor minimum.

Q.42 (3)

Let $f(x)=x^5-5x^4+5x^3-1$

$\Rightarrow f'(x)=5x^4-20x^3+15x^2=0$

$\therefore (x-3)(x-1)=0$ or $x=3,1$

Now $f''(x)=20x^3-60x^2+30x$

Put $x=3$ and 1, we get $f''(3)=+ve$ and $f''(1)=-ve$ and $f''(0)=0$. Hence $f(x)$ neither maximum nor minimum at x = 0

Q.43 (4)

$f'(x)=6x^2-6x-12$

$f'(x)=0 \Rightarrow (x-2)(x+1)=0 \Rightarrow x=-1,2$

Here $f(4)=128-48-48+5=37$

$f(-1)=-2-3+12+5=12$

$f(2)=16-12-24+5=-15$

$f(-2)=-16-12+24+5=1$

Therefore the maximum value of function is 37 at $x=4$.

Q.44 (2)

Let α,β be the roots of the equation

$x^2-(a-2)x-a+1=0,$

then $\alpha+\beta=a-2,\ \alpha\beta=-a+1$

$\therefore z=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$=(a-2)^2+2(a-1)=a^2-2a+2$

$\frac{dz}{da}=2a-2=0 \Rightarrow a=1$

$\frac{d^2z}{da^2}=2>0,$ so *z* has minima at $a=1$

So $\alpha^2+\beta^2$ has least value for $a=1$. This is because we have only one stationary value at which we have minima. Hence $a=1$.

Q.45 (1)

$y=x^5-5x^4+5x^3-10$

$\therefore \frac{dy}{dx}=5x^4-20x^3+15x^2=5x^2(x^2-4x+3)$

$=5x^2(x-3)(x-1)$

$\frac{dy}{dx}=0$, gives $x=0,1,3$

Now, $\frac{d^2y}{dx^2}=20x^3-60x^2+30x=10x(2x^2-6x+3)$

and $\frac{d^3y}{dx^3}=10(6x^2-12x+3)$

For $x=0: \frac{dy}{dx}=0, \frac{d^2y}{dx^2}=0, \frac{d^3y}{dx^3}\neq 0$

$\therefore$ Neither minimum nor maximum

For $x=1, \frac{d^2y}{dx^2}=-10=$ negative.

$\therefore$ Maximum value $y_{max.}=-9$

For $x=3, \frac{d^2y}{dx^2}=90=$ positive

$\therefore$ Minimum value $y_{min.}=-37$

Q.46 (2)

Let $y=\frac{x^2-x+1}{x^2+x+1}$

$\Rightarrow \frac{dy}{dx}=\frac{(x^2+x+1)(2x-1)-(x^2-x+1)(2x+1)}{(x^2+x+1)^2}$

$\Rightarrow \frac{dy}{dx}=\frac{2x^2-2}{(x^2+x+1)^2}=0 \Rightarrow 2x^2-2=0 \Rightarrow x=-1,+1$

$\frac{d^2y}{dx^2}=\frac{4(-x^3+3x+1)}{x^2+x+1}$

At $x=-1$, $\frac{d^2y}{dx^2}<0,$ the function will occupy maximum value, $\therefore f(-1)=3$ and at x = 1 $\frac{d^2y}{dx^2}>0,$ the function will occupy minimum value $\therefore f(1)=\frac{1}{3}$.

Q.47 (4)

Let $y=\frac{\log x}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{x.\frac{1}{x}-\log x}{x^2}=\frac{1-\log x}{x^2}=\frac{1-\log x}{x^2}$

Put $\frac{dy}{dx}=0 \Rightarrow \frac{1-\log x}{x^2}=0$

$\Rightarrow 1-\log x=0 \Rightarrow x=e$ and

$\frac{d^2y}{dx^2}=\frac{-3x+2x\log x}{x^4}$

At $x=e$, $\frac{d^2y}{dx^2}=\frac{1}{-e^3}<0$

$\therefore$ In $[2,\infty)$ the function $\frac{\log x}{x}$ will be maximum and minimum value does not exist.

Q.48 (3)

Let the positive number $\left(x+\frac{1}{x}\right)$ will be minimum,

when $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}>0$.

Differentiate with respect to *x*, we have $1-\frac{1}{x^2}=0$

$\Rightarrow x=-1,1$ and $\frac{d^2y}{dx^2}=+\frac{2}{x^3} \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=1}>0$

So, at $x=1,\left(x+\frac{1}{x}\right)$ will be minimum.

Q.49 (3)

$f(x)=\frac{x}{4+x+x^2}$

Differentiate, $f'(x)=\frac{4+x+x^2-x(1+2x)}{(4+x+x^2)^2}$

For maximum $f'(x)=0 \Rightarrow \frac{4-x^2}{(4+x+x^2)^2}=0$

$\Rightarrow x=2,-2$

Both values of *x* are out of interval

$\therefore f(-1)=\frac{-1}{4-1+1}=\frac{-1}{4}$,

$f(1)=\frac{1}{4+1+1}=\frac{1}{6}$ (maximum).

Q.50 (1)

$f(x)=2x^3-15x^2+36x+4$

$f'(x)=6x^2-30x+36$ …..(i)

We know that for its maximum value $f'(x)=0$. $6x^2-30x+36=0 \Rightarrow (x-2)(x-3)=0$

$\Rightarrow x=2,3$.

Again differentiating equation (i), we get $f''(x)=12x-30$

$\Rightarrow f''(2)=24-30=-6<0$

Therefore $f(x)$ is maximum at $x=2$.

Q.51 (2)

$f(x)=2x^3-3x^2-12x+4$

$f'(x)=6x^2-6x-12$

Now $f'(x)=0 \Rightarrow x^2-x-2=0 \Rightarrow x=2,-1$

Now $f''(x)=12x-6 \Rightarrow f''(2)=+ve$,

$f''(-1)=-ve$

$\therefore$ Given function has one maximum and one minimum.

Q.52 (3)

$f(x)=y=x^{-x} \Rightarrow \log y=-x\log x$

Differentiating *w.r.t. x*, $\frac{1}{y}.\frac{dy}{dx}=-\left[x.\frac{1}{x}+\log x\right]$

$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=-[1+\log x] \Rightarrow \frac{dy}{dx}=-x^{-x}[1+\log x]$

$\Rightarrow \frac{dy}{dx}=x^{-x}\left[\log\frac{1}{x}-1\right]$

Put $\frac{dy}{dx}=0$ $\Rightarrow \log_e\frac{1}{x}=\log_e e$

$\Rightarrow \frac{1}{x}=e \Rightarrow x=\frac{1}{e}$.

Q.53 (1)

$y=a(1-\cos x) \Rightarrow y'=a\sin x$

$\Rightarrow y'=0 \Rightarrow \sin x=0 \Rightarrow x=0,\pi$

Now $y''=a\cos x \Rightarrow y''(0)=a$ and $y''(\pi)=-a$

Hence *y* is maximum when $x=\pi$

Q.54 (1)

$f(x)=2x+3y$ when $xy=6$

$f(x)=2x+3y=2x+\frac{18}{x}$

$f'(x)=2-\frac{18}{x^2}=0$

$\Rightarrow x=\pm 3$ and $f''(x)=\frac{36}{x^3} \Rightarrow f''(3)>0$

Putting $x=+3$, we get the minimum value to be 12.

Q.55 (2)

Let $f(x)=4e^{2x}+9e^{-2x}$

$\therefore f'(x)=8e^{2x}-18e^{-2x}$

Put $f'(x)=0 \Rightarrow 8e^{2x}-18e^{-2x}=0$

$e^{2x}=3/2 \Rightarrow x=\log(3/2)^{1/2}$

Again $f''(x)=16e^{2x}+36e^{-2x}>0$

Now $f(\log(3/2)^{1/2})=4e^{2.(\log(3/2)^{1/2})}+9e^{-2(\log(3/2)^{1/2})}$

$=4\times\frac{3}{2}+9\times\frac{2}{3}=6+6=12$

Hence minimum value = 12.

Q.56 (4) $x+2y=8$, $y=\frac{8-x}{2}$

Now $f(x)=xy=x.\frac{(8-x)}{2}=4x-\frac{x^2}{2}$

$\therefore\ f'(x)=4-x$

For extremum, $f'(x)=0$

$\therefore\ x=4$ and $y=2$.

Also $f''(x)=-1<0$

So, maximum value of $xy=4\times 2=8$

Q.57 (2)

$x+y=20$ and $z=xy^3$

$\Rightarrow\ z=y^3(20-y)=20y^3-y^4$

$\Rightarrow\ \frac{dz}{dy}=60y^2-4y^3=0\ \Rightarrow\ 4y^2(15-y)=0$

So, either $y=0$ or $y=15$

Now, $\frac{d^2z}{dy^2}=120y-12y^2$; $\therefore$ At $y=0, \frac{d^2z}{dy^2}>0$

$\therefore\ y=0$ is the point of minima and at $y=15, \frac{d^2z}{dy^2}<0$

$\therefore\ y=15$ is the point of maxima.

Hence the required parts is (5, 15).

Q.58 (3)

We know that perimeter of a rectangle $S=2(x+y)$, where *x* and *y* are adjacent sides

$\Rightarrow\ y=\frac{S-2x}{2}$.

Now area of rectangle,

$A=xy=\frac{x}{2}(S-2x)=\frac{1}{2}(Sx-2x^2)$

Differentiating *w.r.t. x* of *A*, we get

$\frac{dA}{dx}=\frac{1}{2}(S-4x)=0\ \therefore x=\frac{S}{4}$ and $y=\frac{S}{4}$

Again $\frac{d^2A}{dx^2}=-ve$

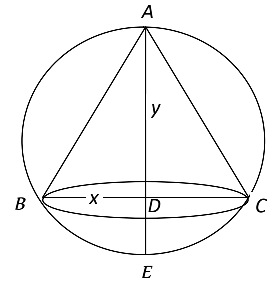
Hence the area of rectangle will be maximum when rectangle is a square.

Q.59 (2)

Let diameter of sphere $AE=2r$

Let radius of cone is *x* and height is *y*

$\therefore\ AD=y$ since $BD^2=AD.DE$



or $x^2=y(2r-y)$ .....(i)

Volume of cone

$V=\frac{1}{3}\pi x^2y=\frac{1}{3}\pi y(2r-y)y=\frac{1}{3}\pi(2ry^2-y^3)$

$\Rightarrow\ \frac{dV}{dy}=\frac{1}{3}\pi(4ry-3y^2)\ \Rightarrow\ \frac{dV}{dy}=0$

$\Rightarrow\ \frac{1}{3}\pi(4ry-3y^2)=0\ \Rightarrow\ y(4r-3y)=0$

$\Rightarrow\ y=\frac{4}{3}r, 0$

Now $\frac{d^2V}{dy^2}=\frac{1}{3}\pi(4r-6y)$, put $y=\frac{4}{3}r$

$\Rightarrow\ \frac{d^2V}{dy^2}=\frac{1}{3}\pi\left(4r-6\times\frac{4}{3}r\right)$ = negative value

So, volume of cone is maximum at $y=\frac{4}{3}r$

$\Rightarrow\ \frac{\text{Height}}{\text{Radius}}=\frac{y}{r}=\frac{4}{3}$

Q.60 (2)

Let $f(x)=x^3-12x^2+36x+17$

$\therefore f'(x)=3x^2-24x+36=0$ at $x=2,6$

Again $f''(x)=6x-24$ is $-ve$ at $x=2$

So that $f(6)=17,\ f(2)=49$

At the end points

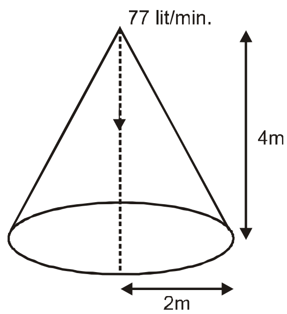
$=f(1)=42,\ f(10)=177$

So that $f(x)$ has its maximum value as 177.

JEE-MAIN
OBJECTIVE QUESTIONS

Q.1 (2)

$V = \frac{1}{3}\pi r^2 h \quad \left(\because \frac{r}{h} = \frac{2}{4} = \frac{1}{2}\right)$



$V = \frac{1}{3}\pi \frac{h^3}{4} = \frac{\pi}{12}h^3$

$\frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$

$77 \times 10^3 = \frac{22}{7} \times \frac{1}{4} \times 70 \times 70 \times \frac{dh}{dt}$

($\because$ 1 litre = 10^3 c.c.)

$\therefore \frac{dh}{dt} = 20$ cm/min.

Q.2 (1)

$V = \pi r^2 h$

$\frac{dv}{dt} = \pi r^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{dv/dt}{\pi r^2} = \frac{1}{9\pi}$ m/min.

Q.3 (1)

$y = \tan(\tan^{-1} x)$

$\Rightarrow y = x$

$\Rightarrow x = -\sqrt{x} + 2$

$x + \sqrt{x} - 2 = 0$

$\sqrt{x} = 1 \Rightarrow x = 1, \; y = 1$

$\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$

$\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{1}{2}$

Slope of normal = 2

Equation of normal is 2x – y =1

Q.4 (3)

$y - e^{xy} + x = 0$

Differentiating w.r.t. to y

$1 - e^{xy}\left(\frac{dx}{dy}.\, y + x\right) + \frac{dx}{dy} = 0, \quad \frac{dx}{dy} = 0$

$1 - xe^{xy} = 0, \; xe^{xy} = 1 \Rightarrow x = 1, y = 0$

$\therefore$ Point is (1, 0)

Q.5 (4)

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(-\sin\theta)}{a(1+\cos\theta)}$

$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{-\sqrt{3}}{3} = -\frac{1}{\sqrt{3}}$

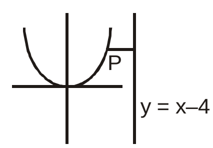
$\tan\alpha = -\frac{1}{\sqrt{3}} \Rightarrow \alpha = \pi - \frac{\pi}{6}$

$\alpha = \frac{5\pi}{6}$

Q.6 (1)

Let the point on parabola P $(2t, t^2)$

$y = \frac{x^2}{4}$



$\frac{dy}{dx} = \frac{2x}{4} = \left.\frac{x}{2}\right|_{(2t, t^2)} = t$

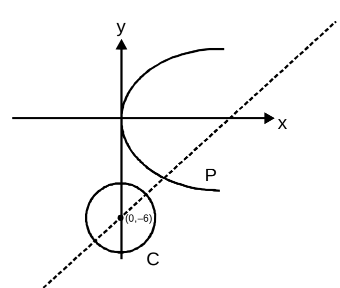
slope = 1 $\Rightarrow$ t = 1 so P(2, 1)

Q.7 (1)

$P_1 : y^2 = 8x$

$C_1 : x^2 + (y + 6)^2 = 1$

$2y\frac{dy}{dx} = 8 \Rightarrow \frac{dy}{dx} = \frac{4}{y}$



Equation of normal of parabola
$y = mx - 2am - am^3$
if passes through $(0,-6)$

$-6 = -2am - am^3$
$\because a = 2 \Rightarrow 3 = 2m + m^3$
$m^3 + 2m - 3 = 0 \Rightarrow m = 1.$
Point on parabola $(am^2, -2am) \equiv (2, -4).$

Q.8 (2)

$y^2 = 4a(x + a \sin \frac{x}{a})$
Let point P (h, k)

$2y\,y' = 4a\left(1+\cos\frac{x}{a}\right)$

slope should be equal to zero

$\cos \frac{x}{a} = -1$

point will lie on the curves also

$k^2 = 4a(h + a \sin \frac{h}{a})\ \sin \frac{h}{a} = 0$

$k^2 = 4ah \Rightarrow y^2 = 4ax$

Q.9 (4)

$x = \sec^2 t$ $\quad$ $y = \cot t$

$\frac{dx}{dt} = 2\sec^2 t \tan t$ $\quad$ $\frac{dy}{dt} = -\operatorname{cosec}^2 t$

$\frac{dy}{dx} = -\frac{\operatorname{cosec}^2 t}{2\sec^2 t \tan t}$

$= -\frac{\cot^2 t}{2\tan t} = -\frac{1}{2\tan^3 t}$

$\frac{dy}{dx} = -\frac{1}{2}$ $\quad$ P(2, 1)

at $t = \frac{\pi}{4}$ $\quad$ $x_1 = \sec^2 \frac{\pi}{4} = 2$

$y_1 = \cot\frac{\pi}{4} = 1$

$y - 1 = -\frac{1}{2}(x - 2)$ curve

$2y - 2 = -x + 2$ $\quad$ $x = 1 + \tan^2 t$

$2y + x = 4$ $\quad$ $x = 1 + \frac{1}{y^2}$

Let $Q(x_1, y_1)$
Solve tangent with curve equation

$4 - 2y = 1 + \frac{1}{y^2}$

$y = 1, -\frac{1}{2}$

$y = -\frac{1}{2} \Rightarrow x = 5$

$Q\left(5, -\frac{1}{2}\right)$

P(2, 1)

$PQ = \frac{3}{2}\sqrt{5}$

Q.10 (2)

$x^2 + y^2 - 2x - 3 = 0$

$2x + 2y\frac{dy}{dx} - 2 = 0$

$\frac{dy}{dx} = \left.\frac{2-2x}{2y}\right|_p = \left.\frac{2-2x_1}{2y_1}\right|_p$

$2 - 2x_1 = 0$
$x_1 = 1$
$1 + y_1^2 - 2 - 3 = 0$
$y_1^2 = 4$
$y_1 = \pm 2$
Two points (1, 2) and (1, –2)

Q.11 (2)

$3x + 4y = c$ $\quad$ $\frac{x^4}{2} = x + y$

$y = -\frac{3}{4}x + \frac{c}{4}$ $\quad$ $\frac{4x^3}{2} = 1 + y'$

$c = 3x_1 + 4y_1$ $\quad$ $y' = 2x^3 - 1$

$= \frac{3}{2} - \frac{60}{32}$ $\quad$ $2x_1^3 - 1 = -\frac{3}{4}$

$2x_1^3 = \frac{1}{4}$

$c = -\frac{12}{32}$ $\quad$ $x_1 = \frac{1}{2}$

unique value of c

$y_1 = \frac{x_1^4}{2} - x_1$

$= \frac{1}{3^2} - \frac{1}{2} = -\frac{15}{32}$

Q.12 (2)

$x^2y = 1 - y$
$xy = 1 - y$
$x^2y = xy$

$x^2y - xy = 0$
$xy(x - 1) = 0$
$xy = 0$ & $x = 1$
$\Rightarrow y = 1/2$
$x = 0$ $\Rightarrow y = 1$
POI P(1, 1/2)
$y = 0$ $0 = 1$ not possible Q(0, 1)
$x^2y = 1 - y$
$2xy + x^2y' = -y'$

$$y' = \left.\frac{-2xy}{1+x^2}\right|_p = \frac{-2(1)(1/2)}{1+1} = -\frac{1}{2}$$

$y'|_Q = 0$

at P tangent $y - \frac{1}{2} = -\frac{1}{2}(x - 1)$ ....(1)

at Q tangent $y - 1 = 0 \Rightarrow y = 1$ ....(2)
Intarsection of (1) & (2)

$$\frac{1}{2} = -\frac{1}{2}(x-1)$$

$x - 1 = -1$
$x = 0$
POI (0, 1)

Q.13 (1)

$$\text{L.N.} = y\sqrt{1+y'^2}$$

$$\text{L.N.} = \frac{y}{y'}\sqrt{1+y'^2}$$

$$\frac{(\text{L.N.})^2}{(\text{L.T})^2} = y'^2$$

Also $\frac{\text{L.S.N}}{\text{L.S.T}} = \frac{yy'}{\frac{y}{y'}} = y'^2$

Q.14 (2)

$y = \frac{a}{2}(e^{x/a} + e^{-x/a})$ P(x_1, y_1)

$$\frac{dy}{dx} = \frac{a}{2}\left.\left(\frac{1}{a}e^{x/a} - \frac{1}{a}e^{-x/a}\right)\right|_p$$

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{x_1/a} - e^{-x_1/a}\right)$$

$$L_N = y_1\sqrt{1+m^2}$$

$$= \frac{a}{2}(e^{x_1 a} + e^{-x_1/a})\sqrt{1+\frac{1}{4}(e^{x_1/a} - e^{-x_1/a})^2}$$

$$a \times L_N = \frac{a}{2}(e^{x_1/a} - e^{-x_1/a}) \quad \frac{a}{2}(e^{x_1/a} + e^{-x_1/a})$$

$a \times L_N = y_1^{\,2}$
$y_1^{\,2} = a \times L_N$
quantity = a

Q.15 (2)
Subtangent = Subnormal
$L_{ST} = L_{SN}$

$$\frac{y_1}{m} = y_1 m$$

$m^2 = 1$

$$L_T = \left|\frac{y_1\sqrt{1+m^2}}{m}\right| = y_1\sqrt{2}$$

$$= \sqrt{2}\,y_1 = \sqrt{2}\text{ ordinate}$$

Q.16 (2)
$x^3 + pxy^2 = -2$; $3x^2y - y^3 = 2$
$3x^2 + P(y^2 + 2xyy') = 0$; $6xy + 3x^2y' - 3y^2y' = 0$

$$m_1 = y' = \frac{3x^2+py^2}{-2pxy},\ m_2 = y' = -\frac{2xy}{3x^2-3y^2}$$

$m_1 \times m_2 = -1$

$$\frac{(3x^2+py^2)}{-2pxy} \times \frac{(-6xy)}{(3x^2-3y^2)} = -1$$

$$\frac{3}{p}\,\frac{(3x^2+py^2)}{(3x^2-3y^2)} = -1$$

p = –3 only possible

Q.17 (2)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} =$$

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$$

$y + xy' = 0$

$m_2 = y' = -\frac{y}{x}$

$$m_1 = y' = \frac{b^2}{a^2}\,\frac{x}{y}$$

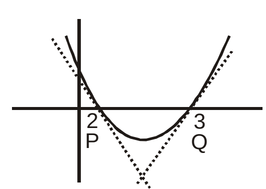
$m_1 \times m_2 = -1$

$$\frac{b^2}{a^2}\,\frac{x_1}{y_1} \times \left(-\frac{y_1}{x_1}\right) = -1$$

$b^2 = a^2$

Q.18 (1)
$y = x^2 - 5x + 6$
$y = (x - 2)(x - 3)$

$$\frac{dy}{dx} = 2x - 5$$



$$m_1 = \left.\frac{dy}{dx}\right|_P = -1$$

$m_2 = \left.\frac{dy}{dx}\right|_Q = 1$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-1-1}{1-1}\right| = \infty$

$\theta = \frac{\pi}{2}$

Q.19 (1)

$\frac{dy}{dx} \le 0 \quad \forall \ x \in R$

$\Rightarrow a + 2 < 0, \qquad D \le 0$

$\Rightarrow a + 2 < 0, \qquad a(a + 3) \ge 0$

$\Rightarrow a \le -3$

Q.20 (3)

Let $z = x^3$

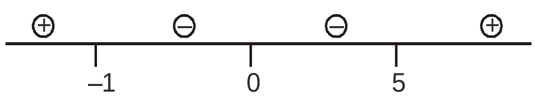
$y = 6x^2 + 15x + 5$

$\frac{dy}{dx} > 1$

$\frac{12x + 15}{3x^2} - 1 > 0$

$\frac{12x + 15 - 3x^2}{3x^2} > 0$

$\frac{x^2 - 4x - 5}{x^2} < 0 \Rightarrow \frac{(x+1)(x-5)}{x^2} < 0$



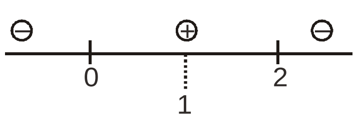
$x \in (-1, 5)$

Q.21 (3)

$f(x) = \frac{|x-1|}{x^2}$

$x \ge 1 \ f(x) = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}$

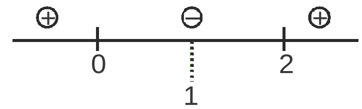
$f'(x) = -\frac{1}{x^2} + \frac{2}{x^3} = -\frac{(x-2)}{x^3}$



$x \in (2, \infty)$ decreasing

$x < 1 \ f(x) = \frac{1-x}{x^2}$

$f'(x) = \frac{x-2}{x^3}$



$x \in (0, 1)$

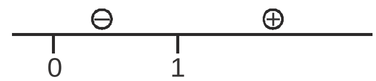
$x \in (0, 1) \cup (2, \infty)$ decreasing

Q.22 (4)

$f(x) = x\ell nx - x + 1 \qquad D_f : x \in R^+$

$f'(x) = \ell nx + 1 - 1$

$x = 1 \qquad$ Critical point



If $x \in (0, 1)$ f is ↓ing

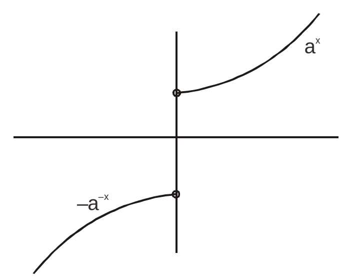
$f(1) > f(x) > f(0) \Rightarrow 0 > f(x) > 1$ positive

If $x \in (1, \infty)$ & is ↑ing

$f(x) > f(1) \Rightarrow \boxed{f(x) > 0}$

Q.23 (4)

$(\ell na)\, h(x) = \ell n\, f(x)\, g(x)$



$= \ell n \left[a^{\{a^{|x|}\text{sgn } x\} + [a^{|x|}\text{sgn } x]}\right]$

$= \ell n\, a^{a^{|x|}\text{sgn} x}$

$(\ell n\, a)\, h(x) = a^{|x|}\, \text{sgn } x\, (\ell n\, a)$

$h(x) = a^{|x|}\, \text{sgn } x$

$h(x) = a^x \qquad x > 0$

$\quad = 0 \qquad x = 0$

$\quad = -a^{-x} \qquad x < 0$

h is odd and ↑ing.

Q.24 (4)

Given that $f(x)\, f'(x) < 0$

$f(x)\, f'(x) < 0$

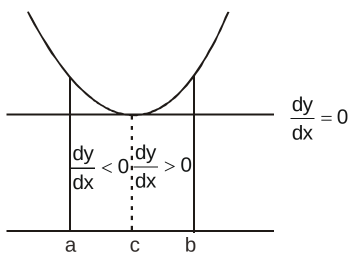
$y = |f(x)| \rightarrow \begin{cases} f(x) & f(x) > 0 \\ -f(x) & f(x) < 0 \end{cases}$

$y' = \begin{cases} f'(x) & f(x) > 0 & ...(1) \\ -f'(x) & f(x) < 0 & ...(2) \end{cases}$

(1) $f(x) > 0$, $f'(x) < 0$ decreasing
(2) $f(x) < 0$, $f'(x) > 0$, but $y' < 0$
so $|f(x)|$ is decreasing function.

Q.25 (3)
For $x \in (a, b)$

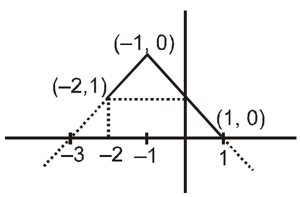
$$\frac{dy}{dx}\uparrow \Rightarrow \frac{d^2y}{dx^2} > 0$$



either $\frac{dy}{dx} < 0 \Rightarrow \frac{d^2y}{dx^2} > 0$

$\frac{dy}{dx} > 0 \Rightarrow \frac{d^2y}{dx^2} > 0$

Q.26 (3)
$f(x) = 2 - |x + 1|$



Figure

From figure it is clear that greatest, least values are respectively 2, 0

Q.27 (4)
$f(1) = 1 - 1 + 10 - 5 = 5$
for greatest value at $x = 1$
$f(1^+) \le f(1)$
$-2 + \log_2 (b^2 - 2) \le 5$;
$\log_2 (b^2 - 2) \le 7$
$b^2 - 2 \le 2^7$
$b^2 \le 130$
$-\sqrt{130} \le b \le \sqrt{130}$

$b^2 - 2 > 0$
$b > \sqrt{2}$ or $b < -\sqrt{2}$

final answer $b \in [-\sqrt{130}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{130}]$

Q.28 (4)

$f(x) = \sin 2x - x$ $\qquad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$f'(x) = 2\cos 2x - 1 = 0$

$\cos 2x = \frac{1}{2}$

$2x = \frac{\pi}{3}, -\frac{\pi}{3}$

$x = \frac{\pi}{6}, -\frac{\pi}{6}$

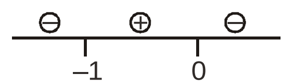
$f\left(\frac{x}{2}\right) = -\frac{\pi}{2}$ $\qquad f\left(-\frac{x}{6}\right) = -\frac{1}{2} + \frac{\pi}{6}$

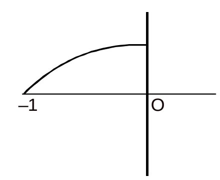
$f\left(-\frac{x}{2}\right) = \frac{\pi}{2}$ $\qquad f\left(\frac{x}{6}\right) = \frac{1}{2} - \frac{\pi}{6}$

$\text{Diff} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

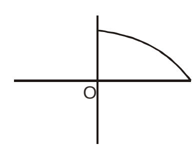
Q.29 (2)
Let $f(x) = \ell n (1 + x) - x$ $\qquad D_f$: $\boxed{x > -1}$

$f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x}$





In $x \in (-1, 0)$ f is $\uparrow$ing
$f(x) \le f(0)$
$f(x) \le 0$
In $x \in (0, \infty)$ f is $\downarrow$ing.



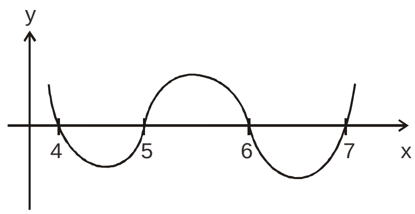
$f(x) \le f(0)$
$f(x) \le 0$

Q.30 (1)
$f(x) = x^3 - 6x^2 + ax + b$
$f(x)$ satisfies condition in Rolle's theorem on [1, 3]
$f(1) = f(3)$
$\Rightarrow 1 - 6 + a + b = 27 - 54 + 3a + b$
$2a = 22$
$a = 11$
and $b \in R$.

Q.31 **(3)**
$f'(x) = 0 \Rightarrow x = -2, 3$
$x = -2 \in (-3, 0)$
$\therefore c = -2$

Q.32 (2)

$f(x) = (x - 4)(x - 5)(x - 6)(x - 7)$



This curve f(x) cutting, y = 0 curve four times then in between [4, 7] there will be three points where $f'(x) = 0$ by rolle's therm.

Alter:

$f'(x) = (x - 5)(x - 6)(x - 7) + (x - 4)(x - 6)(x - 7) + (x - 4)(x - 5)(x - 7) + (x - 4)(x - 5)(x - 6)$

$f'(4) < 0$

$f'(5) > 0$

$f'(6) < 0$

$f'(7) > 0$

by L.M.V.T. we can say that is [4, 5] one root

Q.33 (2)

$f(0) = 2,\ g(0) = 0,\ f(1) = 6,\ g(1) = 2$

(1) $h(x) = f'(x) - g(x)$

$h(0) = f(0) - g(0) = 2$ wrong

$h(1) = f(1) - g(1) = 6 - 2 = 4$

(2) $h(x) = f(x) - 2g(x)$

$h(0) = f(0) - 2g(0) = 2$ right`

$h(1) = f(1) - 2g(1) = 6 - 4 = 2$

Q.34 (2)

$\int_0^1 (3x^2 + 4ax + b)\,dx = 0$ By LMVT

$x^3 + 2ax^2 + bx\,\big|_0^1 = 0 \Rightarrow 1 + 2a + b = 0$

Q.35 (3)

(1) $f(0) = 0$

$f(1) = 0$ Rolle's Thrm. is applicable

(3) $f'(3) = \dfrac{f(3) - f(-3)}{3 + 3}$

$e^c = \dfrac{e^3 - e^{-3}}{6}$

$e^c = \dfrac{e^6 - 1}{6e^3}$

$c = \ell n\left(\dfrac{e^6 - 1}{6e^3}\right) = \ell n(e^6 - 1) - \ell n6 - 3$

Q.36 (4)

(1) LMVT

(2) LMVT

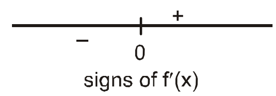
(3) $f(0) = -2$

$f(1) = 4 - 5 + 1 - 2 = -2$

Not applicable

Q.37 (3)

$f'(x) = (2^2 + 4^2x^2 + 6^2x^4 + ... + 100^2x^{98})\,x$



Minimum at x = 0

Q.38 (1)

$f'(x) = \sin x \cos x\,(3\sin x + 2\lambda)$

$f'(x) = 0$

$\Rightarrow \sin x = 0$ or $\cos x = 0$ or $\sin x = \dfrac{-2\lambda}{3}$

$\Rightarrow x = 0$ or $\sin x = \dfrac{-2\lambda}{3}$ (as $\cos x = 0$ is not possible).

If $\lambda = 0$ then $f'(x) \geq 0 \Rightarrow$ no extrema, hence $\lambda \neq 0$

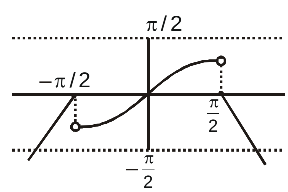
$\Rightarrow -1 < \dfrac{-2\lambda}{3} < 0$ or $0 < \dfrac{-2\lambda}{3} < 1$

$\Rightarrow 0 < \lambda < \dfrac{3}{2}$ or $-\dfrac{3}{2} < \lambda < 0$

Q.39 (2)

$f(x) = \tan^{-1} x,\ |x| < \dfrac{\pi}{2}$

$\dfrac{\pi}{2} - |x|,\ |x| \geq \dfrac{\pi}{2}$

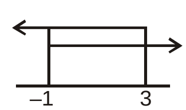


$x = -\dfrac{\pi}{2}$ is maxima

Q.40 (3)

$f(x) = x^3 - 3px^2 + 3(p^2 - 1)x + 1$

$f'(x) = 3x^2 - 6px + 3(p^2 - 1) = 0$



$x = p - 1$ & $p + 1$

$p + 1 > p - 1$

so $p - 1 > -2$ & $p + 1 < 4$

$p > -1$ & $p < 3$

$p \in (-1, 3)$

Q.41 (2)

$f'(x) = \frac{a}{x} + 2bx + 1$

$f'(-1) = 0$

$-a - 2b + 1 = 0$

$a + 2b = 1$

$f'(2) = 0$

$\frac{a}{2} + 4b + 1 = 0 \Rightarrow a + 8b + 2 = 0$

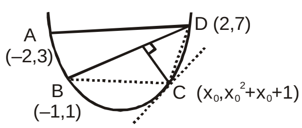
$-6b = 3 \Rightarrow b = \frac{-1}{2}, a = 2$

Q.42 (1)

Let $y = ax^2 + bx + c$

A : $3 = 4a - 2b + c$ ....(1)

B : $1 = a - b + c$ ....(2)



C : $7 = 4a + 2b + c$ ....(3)

$b = 1 = a = c$

$y = x^2 + x + 1$

Method (1) Make determinant using area of ΔBCD then diff with respect to x_0

Method (2) Area will be maximum if tangent at C will be parallel to BD

$\frac{dy}{dx} = 2x_0 + 1 = \left(\frac{7-1}{2+1}\right)$

$2x_0 + 1 = 2$

$x_0 = 1/2$

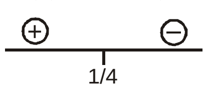
$y = \frac{1}{4} + \frac{1}{2} + 1 = \frac{1+2+4}{4} = \frac{7}{4}$

point $\left(\frac{1}{2}, \frac{7}{4}\right)$

Q.43 (4)

$f(x) = x^{25} (1 - x)^{75}$

$f'(x) = 25x^{24} (1 - x)^{75} - 75x^{25} (1 - x)^{74} = 0$



$\Rightarrow x = 1/4$

$x = 1/4$ maxima

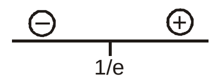
Q.44 (3)

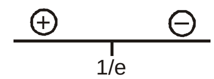
$f(x) = x^x$ | $f(x) = x^{-x}$

$f'(x) = x^x(1 + \ell n\, x)$ | $f'(x) = -x^{-x} (1 + \ell n\, x)$

$1 + \ell n\, x = 0$ | $x = 1/e$

$x = 1/e$





$1/e \rightarrow$ minima

$1/e \rightarrow$ maxima

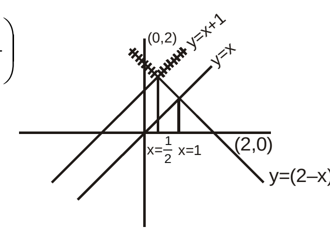
min. value $= \left(\frac{1}{e}\right)^{1/e}$ $f\left(\frac{1}{e}\right) = e^{1/e}$

product $= (e^{-1/e})\,(e)^{1/e} = 1$

Q.45 (4)

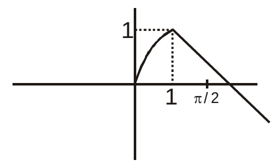
max. value at $\left(x = \frac{1}{2}\right)$

$= 1 + \frac{1}{2} = \frac{3}{2}$



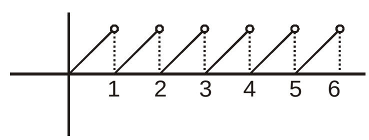
Q.46 (1)

x = 1 local maxima



Q.47 (2)

$f(x) = \{x\}$



x = 5 is minima

Q.48 (2)

$f(x) = x^3 + ax^2 - 9x + b$

$f'(x) = 3x^2 + 2ax - 9$

$f'(1) = 0 \Rightarrow 3 + 2a - 9 = 0 \Rightarrow a = 3$

$f'(x) = 3x^2 + 6x - 9 = 0 \Rightarrow x = 1, x = -3$

$f(1) > 0$ | $f(-3) > 0$

$1 + 3 - 9 + b > 0$ | $-27 + 27 + 27 + b > 0$

$b > 5$ $b > 27$

$\downarrow$

$b > 5$

$b = 6$

$(a, b) = (3, 6)$

Q.49 (2)

$f'(x) = \frac{1}{3(x+1)^{2/3}} - \frac{1}{3(x-1)^{2/3}}$

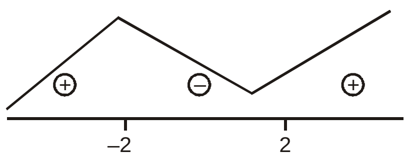
$f'(x) = 0 \quad \Rightarrow x = 0$
$f(0) = 1 + 1 = 2$
$f(1) = 2^{1/3}$
max. value = 2

Q.50 (2)
$f(x) = x^3 - 3x$ $\qquad [0, 2]$
$f'(x) = 3x^2 - 3 = 0 \qquad \Rightarrow x = \pm 1$
$f(1) = 1 - 3 = -2$
$f(-1) = -1 + 3 = 2$ (reject)
$f(0) = 0$
$f(2) = 8 - 6 = 2$
max. value = 2

Q.51 (4)

$$f(x) = \frac{x}{2} + \frac{2}{x}$$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = 0 \quad \Rightarrow x = \pm 2$$



x = 2 is minima
Aliter Approach $\qquad$ AM ≥ GM

$$\frac{\frac{x}{2}+\frac{2}{x}}{2} > \sqrt{\frac{x}{2}\times\frac{2}{x}}, \quad \frac{x}{2} + \frac{2}{x} \geq 2$$

min. value = 2
It is possible only when x = 2

Q.52 (3)
$f(x) = (x - p)^2 + (x - q)^2 + (x - r)^2$
$f'(x) = 2(x - p) + 2(x - q) + 2(x - r) = 0$

$$\Rightarrow x = \frac{p+q+r}{3}$$

$f''(x) = 2 + 2 + 2 > 0$

$x = \frac{p+q+r}{3}$ is minima

Q.53 (2)

Given equation $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$

Let point P (a cos φ, 2 sin φ) on curve
Distance is d form (0, –2)
$z = d^2 = a^2 \cos^2 \phi + 4 (1 + \sin \phi)^2$

$$\frac{dz}{d\phi} = -2 a^2 \cos \phi \sin \phi + 8(1 + \sin \phi) \cos \phi = (4 - a^2) \sin 2\phi + 8 \cos \phi$$

$$\frac{dz}{d\phi} = 0$$

$$\Rightarrow \sin \phi = \frac{4}{a^2 - 4} = \frac{1}{\left(\frac{a^2}{4} - 1\right)} > 1 \text{ as } 1 < \frac{a^2}{4} < 2$$

(reject)

$\cos \phi = 0 \Rightarrow \phi = \frac{x}{2}$

$\frac{d^2z}{d\phi^2} = (4 - a^2)\, 2 \cos 2\phi - 8 \sin \phi$ if $\phi = \frac{x}{2}$ $= 2(a^2 - 8) < 0$

Means $\phi = \frac{\pi}{2}$ will be maxima so point P(0, 2)

Q.54 (3)
$f(x) = 2x^3 - 9ax^2 + 12a^2x + 1 \quad a > 0$
$f'(x) = 6x^2 - 18ax + 12a^2$
$= 6(x^2 - 3ax + 2a^2) = 0$
x = 2a, a
$f''(x) = 6(2x - 3a) |_{x = 2a} = 6a > 0$
$f''(x) = 6(2x - 3a) |_{x = a} = -6a < 0$
x = 2a is minima = q
x = a is maxima = p
$p^2 = q$
$a^2 = 2a$
a = 0 (reject),
a = 2

Q.55 (4)
$f'(x) > 0 \Rightarrow f(x)$ is increasing.
$f''(x) < 0 \Rightarrow f(x)$ is convex (opening down ward)

Q.56 (4)
$x = 1 \Rightarrow 3 = a + b$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 0 \Rightarrow 3a + b = 0$$

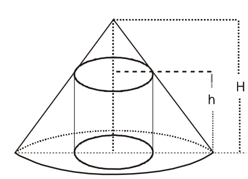
$$a = -\frac{3}{2},\ b = \frac{9}{2}$$

Q.57 (4)

$$\frac{H}{R} = \frac{H-h}{r}$$

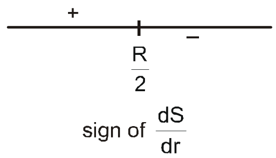
$S = 2\pi rh$

$$= 2\pi H\left(r - \frac{r^2}{R}\right)$$



Figure

$$\frac{dS}{dr} = 2\pi H\left(1 - \frac{2r}{R}\right)$$

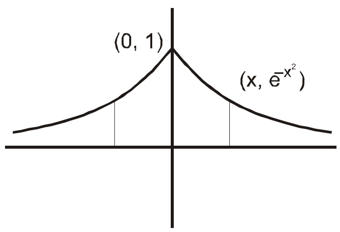


Maximum at $r = \frac{R}{2}$

Q.58 (1)

Let A be area

$A = (2x)\,(e^{-x^2})$, x > 0



Figure

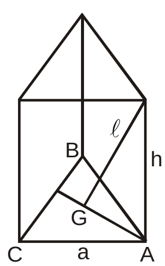
$\frac{dA}{dx} = -2\left(x+\frac{1}{\sqrt{2}}\right)\left(x-\frac{1}{\sqrt{2}}\right)e^{-x^2}$

At $x = \frac{1}{\sqrt{2}}$, A is maximum.

Largest area is $2\frac{1}{\sqrt{2}}\ e^{-1/2}$

Q.59 (2)

$AG = \frac{2}{3}\cdot\frac{a\sqrt{3}}{2} = \frac{a}{\sqrt{3}}$



$\ell^2 = \frac{a^2}{3} + h^2$

$v(h) = 3\,.\,\frac{\sqrt{3}}{4}\,h(\ell^2 - h^2)$,

$v'(h) = 0 \Rightarrow h = \frac{\ell}{\sqrt{3}}$

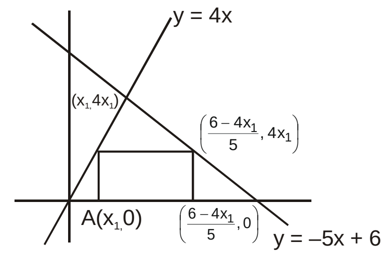
$v_{max} = \frac{\ell^3}{2}$

Q.60 (3)

$A = \left(\frac{6-4x_1}{5} - x_1\right)(4x_1)$

$A = \left(\frac{6-9x_1}{5}\right)(4x_1)$

$\frac{dA}{dx_1} = \frac{4}{5}(6 - 18x_1)$



$\frac{dA}{dx_1} = 0 \Rightarrow x_1 = \frac{1}{3}$

$A = \frac{4}{3}\left(\frac{1}{3}\right)\left(6 - 9\times\frac{1}{3}\right) = \frac{4}{5}$

JEE-ADVANCED
OBJECTIVE QUESTIONS

Q.1 (A)

$\because \quad \frac{dr}{dt} = 3,\ \frac{dh}{dt} = -4, \quad r = 4, h = 6$

$\because \quad v = \pi r^2 h$

$\Rightarrow \quad \frac{dv}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$

$\therefore \quad \frac{dv}{dt} = 2\pi\cdot 4\cdot 6\cdot 3 - \pi\cdot 16\cdot 4$

$= 144\pi - 64\pi = 80\pi$ cu m/s.]

Q.2 (C)

The tangent at $(x_1, \sin x_1)$ is $y - \sin x_1 = \cos x_1\,(x - x_1)$

It passes through the origin.

$\sin x_1 = x_1 \cos x_1 = x_1\sqrt{1-\sin^2 x_1}$

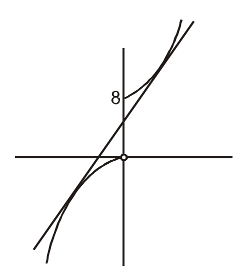
$y_1^2 = \sin^2 x_1 = x_1^2(1 - y_1^2) \Rightarrow (x_1y_1)$ (x_1y_1) lies on the curve

$y^2 = x^2(1 - y^2)$.

$\Rightarrow x^2 - y^2 = x^2y^2$

Q.3 (B)

Let y = mx + c be tangent touching both branches.



$f(x) = -x^2$, $y = mx + c$, x < 0

$x^2 + mx + c = 0$, m > 0 ($\because$ x < 0) (negative roots)

$D = 0 \Rightarrow m^2 = 4c$

$f(x) = x^2 + 8,\ y = mx + c,\ x > 0$

$x^2 - mx + 8 - c = 0$, $m > 0$ (positive roots)

$D = 0 \Rightarrow m^2 = 32 - 4c$

$\Rightarrow c = 4,\ m^2 = 16 \Rightarrow c = 4,\ m = 4$

Q.4 (A)

$$f'(0) = \lim_{x\to 0} \frac{\frac{\sin x^2}{x} - 0}{x - 0} = 1 \text{ (slope of tangent)}$$

slope of normal is – 1

Equation of normal is $y - 0 = -(x - 0)$

Q.5 (D)

$\frac{x}{a} + \frac{y}{b} = 1$ $y = be^{-x/a}$

$\frac{dy}{dx} = -\frac{b}{a}e^{-x/a}$

$\frac{y}{b} = -\frac{y}{b} + 1$ $-\frac{b}{a}e^{-x/a} = -\frac{b}{a}$

$y = -\frac{b}{a}x + b$ $e^{-x/a} = 1$

$x_1 = 0$

$y_1 = b$

Point (0, b)

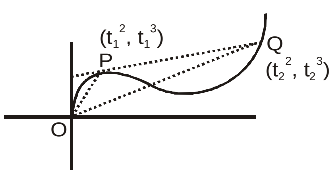
Q.6 (B)

$y^2 = x^3$

Let $P(t_1^2, t_1^3)$

$2y\,y' = 3x^2$

$y' = \frac{3x^2}{2y}$



$y'|_P = \frac{3t_1^4}{2t_1^3} = \frac{3}{2}t_1$

$m_{OP} = \tan\alpha = t_1$

$m_{OQ} = \tan\beta = t_2$

$\frac{\tan\alpha}{\tan\beta} = \frac{t_1}{t_2}$

$M_{PQ} = \frac{t_2^3 - t_1^3}{t_2^2 - t_1^2} = \frac{(t_2 - t_1)(t_2^2 + t_1t_2 + t_1^2)}{(t_2 - t_1)(t_2 + t_1)}$

M_{PQ} = Slope of tangent at point p

$M_{PQ} = y'|_p$

$\frac{t_2^2 + t_1t_2 + t_1^2}{(t_2 + t_1)} = \frac{3}{2}t_1$

$t_1 = -2t_2$

$\frac{t_1}{t_2} = -2$

$\frac{\tan\alpha}{\tan\beta} = -2$

Q.7 (B)

$2y + \frac{dy}{dx} = 4a + 4a\cos\frac{x}{a}$

$\frac{dy}{dx} = 0 \Rightarrow \cos\frac{x}{a} = -1 \Rightarrow \sin\frac{x}{a} = 0$

using given curve

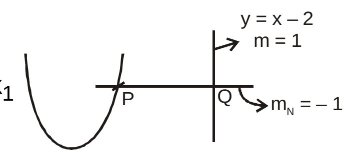
$y^2 = 4a(x + 0)$

$y^2 = 4ax$, a parabola.

Q.8 (B)

$y = x^2$

$\frac{dy}{dx} = 2x\Big|_p = 2x_1$



$2x_1 = 1$

$x_1 = \frac{1}{2}$ $P\left(\frac{1}{2}, \frac{1}{4}\right)$

$y_1 = \frac{1}{4}$

equation of normal

$y - \frac{1}{4} = -\left(x - \frac{1}{2}\right)$

$4y - 1 = -4x + 2$

$4y + 4x = 3$ …(1)

Intersection point

$y = x - 2$ …(2)

$Q\left(\frac{11}{8}, -\frac{5}{8}\right)$

Q.9 (A)

$x^n y = a^n$

$nx^{n-1}y + x^n y^1 = 0$

$y' = -\frac{x^{n-1}y}{x^n}$

$y' = -\frac{y}{x}$

Equation of tangent

$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$

at x-axis $\quad y = 0$

$x_1 = x - x_1 \Rightarrow x = 2x_1$

at y-axis $\quad m = 0$

$y = 2y_1$

Area of $\Delta = \frac{1}{2}(2x_1)(2y_1)$

$= 2x_1y_1$

$= 2x_1 \frac{a^n}{x_1^n}$

$= 2a^n x_1^{(1-n)}$

$1 - n = 0 \Rightarrow n = 1$

Q.10 (B)

$\frac{dy}{dx} = 2x^2 - 4ax + 2 > 0 \ \forall \ x \in R$

$D < 0$

$4^2 a^2 - 4 . 2 . 2. < 0$

$(a + 1)(a - 1) < 0$

$-1 < a < 1$

Q.11 (B)

$y^2 = x^3 + x^2$

curve passes through origin (0, 0)

If we want to drawn the tangent at (0, 0)

$y^2 = x^2$

$y = x$ and $y = -x$

Q.12 (B)

$x = a(\theta + \sin\theta) \quad y = a(1 - \cos\theta)$

$\frac{dy}{dx} = \left.\frac{a\sin\theta}{a(1+\cos\theta)}\right|_{\theta=\frac{x}{2}} = 1 \Rightarrow m = 1$

$x_1 = a\left(\frac{x}{2}+1\right) \quad y_1 = a$

$L_N = y_1\sqrt{1+m^2}$

$L_N = \sqrt{2}\,a$

Q.13 (B)

$y = a^{1-n}x^n$

$\frac{dy}{dx} = n\,a^{1-n}x^{n-1}$

$L_{SN} = |n\,a^{1-n}x_1^{n-1}y_1|$

$= |n\,a^{1-n}x_1^{n-1}a^{1-n}x_1^n|$

$= |n\,a^{2-2n}x_1^{2n-1}|$

$2n - 1 = 0$

$n = 1/2$

Q.14 (D)

For $x^2 = 9a(9 - y)$; $\quad \frac{dy}{dx} = \frac{-2x}{9a} = m_1$

For $x^2 = a(y + 1)$; $\quad \frac{dy}{dx} = \frac{2x}{a} = m_2$

$m_1 m_2 = -1$ gives $4x^2 = 9a^2$ ... (1)

also by eliminating 'y' between the equation of curves $x^2 = 9a$ ...(2)

from (1) and (2), we get $a = 4$]

Q.15 (C)

$y = x^x \ln x$; $\quad y = \frac{2^x - 2}{\ln 2}$

At $x = 1$, $m_1 = \frac{dy}{dx} = x^x.\frac{1}{x} + x^x(\ln x + 1)\ln x = 1.$

At $x = 1$, $m_2 = \frac{dy}{dx} = 2^x\frac{\ln 2}{\ln 2} = 2$

Q.16 (B)

$f'(x) = 3x^2 + 2ax + b + 5\sin 2x \ge 0 \ \forall \ x \in R$

$\because \sin 2x \ge -1$

$\Rightarrow f'(x) \ge 3x^2 + 2ax + b - 5 \ \forall \ x \in R$

$\Rightarrow 3x^2 + 2ax + b - 5 \ge 0 \quad \forall \ x \in R$

$\Rightarrow 4a^2 - 4 .3 . (b - 5) \le 0$

$\Rightarrow a^2 - 3b + 15 \le 0$

Q.17 (B)

$f(x) = \left(\frac{\sqrt{p+4}}{1-p} - 1\right)x^5 - 3x + \ell n5$

$f'(x) = 5\left(\frac{\sqrt{p+4}}{1-p} - 1\right)x^4 - 3$

f '(x) will have critical point have coefficient of x^4 positive. But if the coefficient of x^4 negative and we will not get any critical point and f(x) will be decreasing function. So $\frac{\sqrt{p+4}}{1-p} - 1 \le 0$

If equality holds f '(x) = – 3 is still decreases.

Case-I : $1 - p > 0 \Rightarrow p < 1$

$p + 4 \le (1 - p)^2$

$p^2 - 3p - 3 \ge 0$

$p \in \left(-\infty, \frac{3-\sqrt{21}}{2}\right]$

Case-II : $1 - p < 0 \Rightarrow p > 1$

$$\frac{\sqrt{p+4}}{1-p} \le 1$$

Always true

$p \in (1, \infty)$

$p + 4 \ge 0$

$p \ge -4$

Final answer $p \in \left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$

Q.18 (C)

$x > 1 \Rightarrow f(x) \ge f(1)$

$x > 1 \Rightarrow g(x) \le g(1)$

$\Rightarrow f(g(x)) \le f(g(1))$

$\Rightarrow h(x) \le 1$

.... (i)

Range of h(x) is subset of [1, 10]

$\Rightarrow h(x) \ge 1$

.... (ii)

By (i), (ii) we have $h(x) = 1 \Rightarrow h(2) = 1$

Q.19 (C)

$$f(x) = \frac{x^2}{4}\operatorname{cosec}^2\frac{x}{2}$$

$$f'(x) = \frac{x}{2}\operatorname{cosec}^2\frac{x}{2}\cot\frac{x}{2}\left(\tan\frac{x}{2} - \frac{x}{2}\right)$$

For $0 < x < 1$, $\tan\frac{x}{2} > \frac{x}{2}$

$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is increasing.

Q.20 (C)

$f'(x) = (a^2 - 2a - 2) - \sin x > 0$ or < 0

$a^2 - 2a - 2 > \sin x$ or $a^2 - 2a - 2 < \sin x$

$a^2 - 2a - 2 \ge 1$ or $a^2 - 2a - 2 \le 1$

$a^2 - 2a - 3 \ge 0$ or $a^2 - 2a - 1 \le 0$

$a \ge 3,\ a \le -1$ or $a \in \left[1-\sqrt{2}, 1+\sqrt{2}\right]$

Q.21 (B)

$f(x) = \tan^{-1}(\sin x + \cos x)$

$$f'(x) = \frac{1}{1+(\sin x + \cos x)^2}(\cos x - \sin x) > 0$$

$\cos x - \sin x > 0$

$$\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x > 0$$

$$\cos\left(\frac{\pi}{4} + x\right) > 0$$

$$-\frac{\pi}{2} < \frac{\pi}{4} + x < \frac{\pi}{2} \Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4}$$

Q.22 (D)

$$f(x) = \frac{a\sin x + b\cos x}{c\sin x + d\cos x}$$

$$f'(x) = \frac{ad\cos^2 x + ad\sin^2 x - bc\sin^2 x - bc\cos^2 x}{(c\sin x + d\cos x)^2}$$

$$f'(x) = \frac{ad - bc}{(c\sin x + d\cos x)^2} \ge 0 \Rightarrow ad \ge bc$$

Q.23 (C)

$f(x) = 8ax - a\sin 6x - 7x - \sin x$

$f'(x) = 8a - 6a\cos 6x - 7 - 5\cos 5x$

for $a = 0$ $\quad f'(x) < 0$

for $a = -6$ $\quad f'(x) < 0$

for $a = 6$ $\quad f'(x) > 0$

Q.24 (B)

$f'(x) = 4ax^3 + 3bx^2 + 2x + 1$

$f''(x) = 12ax^2 + 6bx + 2$

$D = 36b^2 - 96 < 0$ (given)

$f'''(x) > 0 \Rightarrow f'(x)$ is increasing

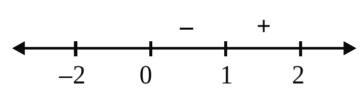
Q.25 (A)

Put $|x| = t$

As, $x \in [-1, 2] \Rightarrow t \in [0, 2]$

let $g(t) = 2t^3 + 3t^2 - 12t + 1$

$\therefore$ $g'(t) = 6(t + 2)(t - 1)$



Sign of g'(t)

$\therefore$ $g(0) = 1$

$g(1) = -6$

and $g(2) = 5$, is the greatest value of function.

Q.26 (D)

$f(1^-) \le f(1)$ and $f(1^+) \le f(1)$

$-2 + \log_2(b^2 - 2) \le 5$

$0 < b^2 - 2 \le 128$ $\quad 2 < b^2 \le 130$

Q.27 (D)

$h(x) = f(x) - g(x) = 1 + x\ \ln\left(x + \sqrt{x^2+1}\right) -$

$\sqrt{x^2+1}$

$h'(x) = \ln\left(x+\sqrt{x^2+1}\right) + \dfrac{x}{\left(x+\sqrt{x^2+1}\right)}$

$\dfrac{(1+x)}{\sqrt{x^2+1}} - \dfrac{x}{\sqrt{x^2+1}}$

$\ln\left(x+\sqrt{x^2+1}\right) = 0 \Rightarrow x = 0$

h(x) is positive
$h(x) \geq h(0)$
$h(x) \geq 0$
$f(x) \geq g(x)$

Q.28 (A)
$f(x) = 1 + x^m (x-1)^n$
$f'(x) = = mx^{m-1}(x-1)^n + nx^m (x-1)^{n-1} = 0$

$\dfrac{m}{x} + \dfrac{n}{(x-1)} = 0$

$mx - m + nn = 0 \Rightarrow x = \dfrac{m}{m+n}$

Q.29 (C)
$f(x) = x(x+3)e^{-x/2}$
$f(x) = (x^2+3x)e^{-x/2}$
$f(-3) = 0 = f(0)$ Rolle's theorem. is applicable

$f'(x) = (x^2+3x)e^{-x/2}\left(-\dfrac{1}{2}\right) + e^{-x/2}(2x+3) = 0$

$x = 3, -2$
$x = -2 \in [-3, 0]$

Q.30 (D)
Let $g(x) = x f(x)$, $g(0) = g(1)$
from Rolle's theorem $g'(c) = 0$
$c f'(c) + f(c) = 0$ **Ans.**]

Q.31 (C)
By Rolle's theorem
$f(1) = f(3) \Rightarrow a + b + 11 - 6$
$= 27a + 9b + 33 - 6$
$\Rightarrow 13a + 4b + 11 = 0$ .....(1)
Now $f'(x) = 3ax^2 + 2bx + 11$

$f'\left(2+\dfrac{1}{3}\right) = 0 \Rightarrow$

$3a\left(2+\dfrac{1}{\sqrt{3}}\right)^2 + 2b\left(2+\dfrac{1}{\sqrt{3}}\right) + 11 = 0$

$\Rightarrow 3a\left(4+\dfrac{1}{3}+\dfrac{4}{\sqrt{3}}\right) + 4b + \dfrac{2b}{\sqrt{3}} + 11 = 0$

$\Rightarrow 13a + \dfrac{12a}{\sqrt{3}} + 4b + \dfrac{2b}{\sqrt{3}} + 11 = 0$

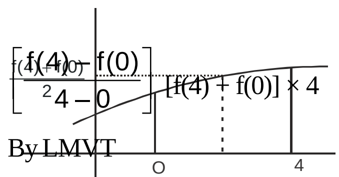
From equation (1), we get

$\dfrac{12a}{\sqrt{3}} + \dfrac{2b}{\sqrt{3}} = 0 \Rightarrow 6a + b = 0$ .....(2)

from equation (1) and (2), we get
$a = 1, b = -6 \Rightarrow a - b = 1 + 6 = 7$ **Ans.**

Q.32 (A)
$f^2(4) - f^2(0)$

$\left[\dfrac{f(4)-f(0)}{4-0}\right][f(4)+f(0)] \times 4$



By LMVT

$f'(a)\left[\dfrac{f(4)+f(0)}{2}\right] \times 8$

$f'(a).f(b) \times 8 = 8 f'(a) f(b)$ or $8f'(b) f(a)$

Q.33 (B)
$f(x) = \tan x$

$f'(C) = \sec^2 C = \dfrac{f(b)-f(a)}{b-a}$

$\sec^2 C = \dfrac{\tan b - \tan a}{b-a}$ $\quad \sec^2 C \geq 1$

for $\left[0, \dfrac{\pi}{2}\right]$

$f(a, b) = \dfrac{\tan b - \tan a}{b-a} \geq 1$

Q.34 (B)
$f(x) = x^3 - 6ax^2 + 5x$
$f'(x) = 3x^2 - 12ax + 5$
By LMVT

$f'(x) = \dfrac{f(2)-f(1)}{2-1}$

$f'(x)\Big|_{x=\frac{7}{4}} = f(2) - f(1)$

$3\left(\dfrac{7}{4}\right)^2 - 12a\left(\dfrac{7}{4}\right) + 5$

$= (8 - 24a + 10) - (1 - 6a - 5)$

$a = \frac{35}{48}$

Q.35 (B)

differentiating both sides, we get

$3f(3x) - f(x) = f(x)$

$\Rightarrow 3f(3x) = 2f(x)$

putting x = 3, $f(9) = \frac{2f(3)}{3} = 2$

According to LMVT,

$f'(c) = \frac{f(9) - f(3)}{9-3} = -\frac{1}{6}$

Q.36 (C)

$f'(x) = 3x^2 - 3p2x + 3p^2 - 3$

$= 3((x-p)^2 - 1)$

$= 3(x - (p+1))(x - (p-1))$

$\Rightarrow p - 1 > -2$ and $p + 1 < 4$

$\Rightarrow p > -1$ and $p < 3$

$\Rightarrow -1 < p < 3$

Q.37 (C)

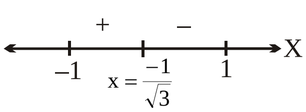
We have $f(x) = \cos^{-1}x + \cos^{-1}x^2$, $x \in [-1, 1]$

$\therefore \quad f'(x) = -\left(\frac{1}{\sqrt{1-x^2}} + \frac{2x}{\sqrt{1-x^4}}\right)$

$= \frac{-1}{\sqrt{1-x^2}}\left(1 + \frac{2x}{\sqrt{1+x^2}}\right)$

Clearly f(x) is decreasing for $x \geq 0$.

Also, $f'(x) = 0$ has only one root $x = \frac{-1}{\sqrt{3}}$.



(local maximum)
Sign scheme of f '(x)

$\Rightarrow$ f (x) has only one local maxima. Ans.]

Q.38 (C)

$f(x) = 2bx^2 - x^4 - 3b$

$f'(x) = 4bx - 4x^3 = 0 \Rightarrow x = 0$ & $x^2 = b$

$f''(x) = 4b - 12x^2 \,|_{x^2=b} = 4b - 12b = -8b < 0$

so f(x) will be max at $x^2 = b$

$g(b) = 2b(b) - b^2 - 3b$

$g(b) = b^2 - 3b$

min. $g(b) = -\frac{D}{4a} = -\frac{9}{4}$

Q.39 (D)

$f(x) = x^p(1-x)^q$

$f'(x) = px^{p-1}(1-x)^q - qx^p(1-x)^{q-1}$

$= \frac{px^p}{x}(1-x)^q - qx^p\frac{(1-x)^q}{(1-x)}$

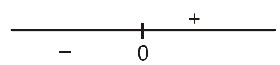
$= x^p(1-x)^q\left[\frac{p}{x} - \frac{q}{1-x}\right] = 0$

$p - px - q - x = 0 \Rightarrow x = \frac{p}{p+q}$

Q.40 (D)

$f'(x) = \frac{2x.\ell n2.2^{x^2}(2^{x^2}+1-\sqrt{2})(2^{x^2}+1+\sqrt{2})}{(2^{x^2}+1)^2}$

$2^{x^2} \geq 1$



sign of f′(x)

$2^{x^2} + 1 - \sqrt{2} \geq 2 - \sqrt{2} > 0$

At x = 0, f(x) is least.

Least value = f(0) = 1

Q.41 (B)

$f'(x) = 3x^2 + 2ax + a = 0 \begin{cases}\alpha \\ \beta\end{cases}$;

$\therefore \quad \alpha + \beta = -\frac{2a}{3}$ and $\alpha\beta = \frac{a}{3}$

given $f(\alpha) + f(\beta) = 2$

$(\alpha^3 + a\alpha^2 + a\alpha + 1) + (\beta^3 + \alpha\beta^2 + \alpha\beta + 1) = 2$

$(\alpha^3 + \beta^3) + a(\alpha^2 + \beta^2) + a(\alpha + \beta) = 0$

$(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) + a[(\alpha + \beta)^2 - 2\alpha\beta] + a(\alpha + \beta) = 0$

$-\frac{8a^3}{27} - a\left(-\frac{2a}{3}\right) + a\left[\frac{4a^2}{9} - \frac{2a}{3}\right] - \frac{2a^2}{3} = 0$

$\frac{4a^3}{27} - \frac{2a^2}{3} = 0 \qquad (a \neq 0 \text{ think !})$

$\frac{4a}{27} = \frac{2}{3} \quad \Rightarrow \quad a = \frac{9}{2}$ Ans.

Q.42 (C)

$y = -x^3 + 3x^2 + 2x - 27$

$y' = -3x^2 + 6x + 2$

max. slope $= -\dfrac{D}{4a} = -\dfrac{(36+24)}{-12} = \dfrac{60}{12} = 5$

Q.43 (C)

$f(x) = x^4 + ax^3 + \dfrac{3x^2}{2} + 1$

$f'(x) = 4x^3 + 3ax^2 + 3x$

$f''(x) = 12x^2 + 6ax + 3$

$= 3(4x^2 + 2ax + 1)$

for concave upward

$f''(x) \geq 0$

$4x^2 + 2ax + 1 \geq 0$

$4a^2 - 16 \leq 0$

$a^2 - 4 \leq 0$

$-2 < a \leq 2$

Q.44 (A)

Let $f(x) = x^3 + px^2 + qx + r$

$\therefore$ $f'(x) = 3x^2 + 2px + q$

Disc. $= 4p^2 - 12q = 4(p^2 - 3q) = 4(p^2 - 2q - q)$

$\therefore$ If $p^2 < 2q \Rightarrow p^2 < 3q$

So, the equation $f(x) = 0$ has one real and two imaginary roots.

Q.45 (B)

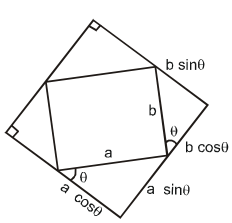
let $f(x) = xe^x$

$f'(x) = (x + 1)e^x$

Q.46 (B)

Area $= ab + \left(\dfrac{1}{2}a^2 \sin\theta \cos\theta + \dfrac{1}{2}b^2 \sin\theta\cos\theta\right).2$

$= ab + \dfrac{(a^2+b^2)}{2} \sin 2\theta$



Figure

Maximum area is $ab + \dfrac{(a^2+b^2)}{2}$

Q.47 (D)

Let point C $(a\cos\theta, b\sin\theta)$

$A = \dfrac{1}{2}\begin{vmatrix} 3 & 0 & 1 \\ 1 & 4 & 1 \\ a\cos\theta & b\sin\theta & 1 \end{vmatrix}$

$A = 6 - b\sin\theta - 2a\cos\theta$

$\dfrac{dA}{d\theta} = 0 \Rightarrow \tan\theta = \dfrac{b}{2a} = \dfrac{1}{\sqrt{2}}$

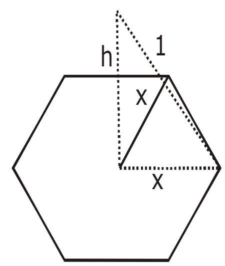
$\Rightarrow a\cos\theta = -\sqrt{6}$ and $b\sin\theta = -\sqrt{6}$

Point C $(-\sqrt{6}, -\sqrt{6})$

Q.48 (C)

$x^2 + h^2 = 1$

volume



$v = \dfrac{1}{3} \times \text{base} \times \text{height}$

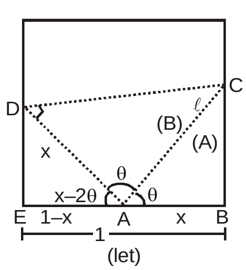
$= \dfrac{1}{3} \times h \times 6 \times \dfrac{\sqrt{3}}{4} x^2$

$v = \dfrac{\sqrt{3}}{2} h(1 - h^2)$

$\dfrac{dv}{dh} = 0 \Rightarrow h = \dfrac{1}{\sqrt{3}}$

Q.49 (B)

A & B triangle are similar



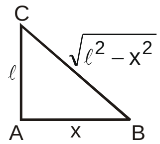
$\cos(x - 2\theta) = \dfrac{1-x}{x} = -\cos 2\theta$

$\cos\theta = \dfrac{x}{\ell}$

$1 - 2\cos^2\theta = \frac{1-x}{x}$

$1 - 2\left(\frac{x^2}{\ell^2}\right) = \frac{1-x}{x}$

$\Delta ABC = \frac{1}{2} x \sqrt{\ell^2 - x^2}$



$= \frac{1}{2}\left(\frac{x}{\ell}\right)\sqrt{1 - \left(\frac{x^2}{\ell^2}\right)}$

$= \frac{1}{2}\left(\frac{x}{\ell}\right)\sqrt{1 - \frac{1}{2}\left(1 - \frac{1-x}{x}\right)} = \frac{1}{2}\left(\frac{x}{\ell}\right)\sqrt{1 + \frac{1}{2x}}$

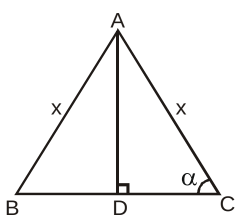
$A = \frac{1}{2}\sqrt{\frac{1}{2}\left(1 - \frac{1-x}{x}\right)}\ \sqrt{1 + \frac{1}{2x}}$

$\frac{dA}{dx} = 0 \Rightarrow x = 2/3$

Q.50 (B)

In ΔACD

$\cos\alpha = \frac{CD}{x}$



$AD = x \sin\alpha$

Area $(\Delta ABC) = 2 \cdot \frac{1}{2}$ (AD) (CD)

$= (x \sin\alpha)(x\cos\alpha)$

$A = \frac{x^2}{2} \sin 2\alpha$

It will max. when $\sin 2\alpha = 1$

$A = \frac{x^2}{2}$

JEE-ADVANCED
MCQ/COMPREHENSION/COLUMN MATCHING

Q.1 (AC)

$y = x^2 + 6x + 10$ $\qquad$ $y = ax^2 + bx + 7/2$

$\frac{dy}{dx} = 2x + 6\Big|_{(-2,\,2)} = 2$; $\frac{dy}{dx} = 2ax + b\Big|_{(1,\,2)}$

$y - 2 = -2(x + 2)$ $\qquad$ $\frac{dy}{dx} = 2a + b$

$y - 2 = -2x + 4$ $\qquad$ $2a + b = -2$ ....(1)

$y + 2x = 6$ $\qquad$ (1, 2) will lie on the curve

$2 = a + b + 7/2$

$a + b = -\frac{3}{2}$ ....(2)

from (1) & (2)

$a = 1,\ b = -5/2$

Q.2 (A,D)

$2y = x^2$ $\qquad$ $P(x_1, y_1)$

$\frac{dy}{dx} = \frac{2x}{2} = x\Big|_p = x_1$

Equation of normal

$y - y_1 = -\frac{1}{x_1}(x - x_1)$

If will pass through (0, 3)

$3 - y_1 = -\frac{1}{x_1}(0 - x_1)$

$3 - y_1 = 1$ $\qquad$ $x_1^2 = 4$

$y_1 = 2$ $\qquad$ $x_1 = 2, -2$

point (2, 2), (–2, 2)

Q.3 (ABC)

$x = t^2 + 3t - 8$ $\qquad$ $y = 2t^2 - 2t - 5$

at point (2, 1)

$2 = t^2 + 3t - 8$ $\qquad$ $-1 = 2t^2 - 2t - 5$

$t = -5, 2$ $\qquad$ $2t^2 - 2t - 4 = 0$

$t = 2$ $\qquad$ $t = 2$ $\quad$ $t = -1$

$x_1 = 4 + 6 - 8$ $\qquad$ $y_1 = 8 - 4 - 5$

$x_1 = 2$ $\qquad$ $y_1 = -1$

$\frac{dy}{dx} = 2t + 3$ $\qquad$ $\frac{dy}{dt} = 4t - 2$

$\frac{dy}{dx} = \frac{4t-2}{2t+3}\Big|_{t=2} = \frac{6}{7}$

$m = 6/7$

$L_T = \left|\frac{y_1\sqrt{1+m^2}}{m}\right| = \left|\frac{-1\sqrt{1+\frac{36}{49}}}{6/7}\right| = \left|\frac{\sqrt{85}}{6}\right|$

Slope of tangent = 6/7

$L_{ST} = \left|\frac{y_1}{m}\right| = \frac{7}{6}$

Q.4 (A,B)

$f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$

$f'(x) = x^2 - 5x + 7 |_P$ $\qquad$ $P(x_1, y_1)$

$f'(x) = x_1^2 - 5x_1 + 7$

cuts equal intercepts that means slope = 1

$x_1^2 - 5x_1 + 7 = 1$

$x_1^2 - 5x_1 + 6 = 0$

$x_1 = 2, \quad x_1 = 3$

$y_1 = 8/3, \qquad y_1 = 7/2$

Point (2, 8/3) $\qquad$ (3, 7/2)

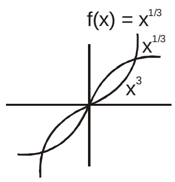
Q.5 (A,B,D)

Tangent is y-axis at (0, 0)

$\Rightarrow x = 0$

Normal $y = 0$

Exactly three points



Q.6 (A,B)

$y = \cos(x + y)$

$y' = -\sin(x + y)(1 + y')$

$y' = -\frac{\sin(x+y)}{1+\sin(x+y)} = -\frac{1}{2}$

$\sin(x + y) = 1$

$\cos(x + y) = 0$

$y_1 = \cos(x_1 + y_1) = 0$

$\sin(x_1 + y_1) = 1 \Rightarrow \sin x_1 = 1$

$x_1 = \frac{\pi}{2}, -\frac{3\pi}{2}$

Point $\left(\frac{\pi}{2}, 0\right)$ and $\left(-\frac{3\pi}{2}, 0\right)$

This two points satisfies

Q.7 (A,C)

$x = a(\cos\theta + \theta\sin\theta)$ $\qquad$ $y = a(\sin\theta - \theta\cos\theta)$

$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)$

$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta)$

$\frac{dy}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$

Normal

$y - a(\sin\theta - \theta\cos\theta) = -\frac{1}{\tan\theta}(x - a(\cos\theta + \theta\sin\theta))$ .....(1)

$y - a(\sin\theta - \theta\cos\theta) = -\cot\theta(x - a(\cos\theta + \theta\sin\theta))$

$= \tan\left(\frac{\pi}{2} + \theta\right)(x - a(\cos\theta + \theta\sin\theta))$

make are angle of $\left(\frac{\pi}{2} + \theta\right)$

Distance from origin of normal (1)

$$d = \left|\frac{-a(\sin\theta - \theta\cos\theta) - a(\cos\theta + \theta\sin\theta)\frac{\cos\theta}{\sin\theta}}{\sqrt{1 + 1/\tan^2\theta}}\right|$$

= a which is constant

Q.8 (A,B)

$by = -ax - c$ $\qquad$ $xy = 2$

$y = -\frac{a}{b}x - c$ $\qquad$ $y + xy' = 0$

$y' = -\frac{y}{x}$

Slope of normal $= \frac{x}{y}$

$\frac{x_1^2}{2} = -\frac{a}{b}$ $\qquad$ $= \frac{x_1}{y_1}$

$= \frac{x_1^2}{2} > 0$

LHS always positive

so RHS should be positive so a, b should have opposita sign

Q.9 (A,C)

$\frac{x^2}{a^2} + \frac{y^2}{4} = 1;$ $\qquad$ $y^3 = 16x$

$\frac{2x}{a^2} + \frac{2yy'}{4} = 0$ $\qquad$ $3y^2y' = 16$

$y' = \left.\frac{-4x}{a^2y}\right|_p$ $\qquad$ $y' = \left.\frac{16}{3y^2}\right|_p$

$m_1 = -\frac{4x_1}{a^2y_1}$ $\qquad$ $m_2 = y' = \frac{16}{3y_1^2}$

$m_1 \times m_2 = -1$

$\frac{-4x_1}{a^2y_1} \times \frac{16}{3y_1^2} = -1$

$\frac{4}{3a^2} = 1$

$a^2 = \frac{4}{3}$

$a = \pm\frac{2}{\sqrt{3}}$

Q.10 (A,C)

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2 \qquad (n \in N)$$

$$x = a \qquad 1 + \left(\frac{y}{b}\right)^n = 2$$

$y^n = b^n$

If n is even $\quad y = \pm b$

If n is odd $\quad y = b$

$n \to$ even (a, b) & (a, –b)

$n \to$ odd (a, b)

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$$

$$n\left(\frac{x}{a}\right)^{n-1}\left(\frac{1}{a}\right) + n\left(\frac{y}{b}\right)^{n-1}\left(\frac{1}{b}\right)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^n}{a^n}\,\frac{x^{n-1}}{y^{n-1}}$$

$n \to$ even

$$(a, b) \Rightarrow \frac{dy}{dx} = -\left(\frac{b}{a}\right)^n \left(\frac{a}{b}\right)^{n-1} = -\frac{b}{a} = m_T$$

$$M_N = \frac{a}{b}$$

$$y - b = \frac{a}{b}(x - a)$$

$by - b^2 = ax - a^2$

$ax - by = a^2 - b^2$

$$(a, -b)\quad \frac{dy}{dx} = -\left(\frac{b}{a}\right)^n \left(-\frac{a}{b}\right)^{n-1} = \frac{b}{a} = m_T$$

$M_N = -a/b$

$$y + b = -\frac{a}{b}(x - a)$$

$ax + by = a^2 - b^2$

Q.11 (A, D)

$y = x^2 + ax + b \qquad$ (1, 0) satisfies

$y = x(c - x) \qquad 1 + a + b = 0$

$= cx - x^2 \qquad c - 1 = 0 \Rightarrow c = 1$

$$m_1 = \frac{dy}{dx} = 2x + a\big|_{(1,\,0)} = 2 + a$$

$$m_2 = \frac{dy}{dx}\ c - 2x\big|_{(1,\,0)} = c - 2$$

$2 + a = c - 2$

$c = 1 \qquad a = -3$

$a + b = -1$

$b = -1 - a$

$b = -1 + 3 = 2$

$b + c = 3$

Q.12 (A,B)

$x = 2 \ln \cot t + 1 \qquad y = \tan t + \cot t$

$$\frac{dx}{dt} = \frac{2}{\cot t}(-\operatorname{cosec}^2 t) \qquad \frac{dy}{dt} = \sec^2 t - \operatorname{cosec}^2 t$$

$$\text{at } t = \frac{\pi}{4}$$

$$\frac{dx}{dt} = -\frac{2}{\sin^2 t}\tan t \qquad \frac{dy}{dt} = 2 - 2 = 0$$

$$= \frac{-2}{(1/2)} = -4$$

$$\frac{dy}{dx} = 0$$

Tangent is parallel to x-axis

Normal is parallel to y-axis

Q.13 (B, C)

$y = ke^{kx} \qquad$ at y-axis

$x = 0,\ y = k$

$$m = \frac{dy}{dx} = k^2 e^{kx}$$

$m = k^2$

$$\tan\theta = \frac{1}{k^2}$$

$$\theta = \tan^{-1}\left(\frac{1}{k^2}\right) = \cot^{-1} k^2 = \sec^{-1}\left(\frac{1}{\sqrt{1+k^4}}\right)$$

Q.14 (A,C,D)

(A) $\quad y^2 = 4ax \quad y = e^{-x/2a}$

$$m_1 = \frac{4a}{2y} = \frac{2a}{y_1}\ ;\ m_2 = -\frac{1}{2a}e^{-x/29}$$

$$= -\frac{1}{2a}y_1$$

$m_1 \times m_2 = -1$

(B) $\quad y^2 = 4ax \quad x^2 = 4ay$

$$y' = \frac{4a}{2y} \qquad 2x = 4ay'$$

$$y' = \frac{2x}{4a}$$

(C) $\quad xy = a^2 \qquad x^2 - y^2 = b^2$

$y + xy' = 0 \qquad 2x - 2yy' = 0$

$$y' = -\frac{y}{x} \qquad y' = \frac{x}{y}$$

$m_1 \times m_2 = -1$

(D) $\quad y = ax \qquad x^2 + y^2 = c^2$

$y' = a \qquad 2x + 2yy' = 0$

$$m_1 \times m_2 = -\frac{ax}{y} = -1 \qquad y' = -\frac{x}{y}$$

Q.15 (A,C)

y = f(x)

Let f(x) = $ax^2 + bx + c$

this parabola touches y = x line at (1, 1)

that means slope at (1, 1) = 1

$f'(x) = 2ax + b$

$2a + b = 1$ ....(1)

(1, 1) will also satisfied the curve

$1 = a + b + c$ ....(2)

$f'(1) = 1$

$2f(0) = 1 - f'(0)$

$f(0) = c$

LHS = 2c

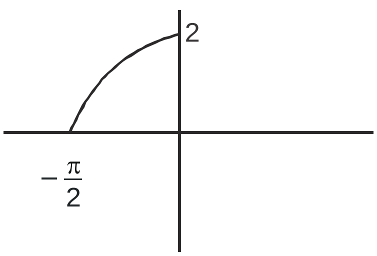
RHS = $1 - f'(0)$ from (1) & (2) a = c

$= 1 - b$

$= 2a = 2c$

Q.16 (A, B)

$f(x) = x + \sin x$



$f'(x) = 1 + \cos x$

$-1 \le \cos x \le 1$

$0 \le 1 + \cos x \le 2$

Increasing

Q.17 (A, B)

$f(x) = x^{m/n}$

Let assume m = 2, n = 3

$f(x) = x^{2/3}$

$f'(x) = \frac{2}{3} x^{-1/3}$ Θ ⊕ (sign change at 0)

f(x) is ↓ is $x \in (-\infty, 0)$

↑ in $x \in (0, \infty)$

Q.18 (A, D)

h(x) = f(x) g(x)

$\Rightarrow f(x)\, g'(x) + g(x)\, f'(x) > 0$

+ve + – –

h is increasing

$h(x) = fog(x) \Rightarrow h'(x) = f'(g(x).\, g'(x) < 0$

h is decreasing

Q.19 (B,C)

$f(x) = 2\ell n\,(x - 2) - x^2 + 4x + 1$

$f'(x) = \frac{2}{x-2} - 2x + 4$

$= 2\left[\frac{1}{(x-2)} - (x-2)\right]$

x = 2, 3, 1 critical points

⊕ ⊖ ⊕ ⊖

↑in $x \in (-\infty, 1) \cup (2, 3)$

Q.20 (A,D)

$f(x) = 2x + \cos^{-1} x + \log\left(\sqrt{1+x^2} - x\right)$

$f'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{(\sqrt{1+x^2} - x)}\left(\frac{2x}{2\sqrt{1+x^2}} - 1\right)$

$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$

$= \frac{2(1+x^2) - 1 - \sqrt{1+x^2}}{(1+x^2)}$

$= \frac{(1+x^2) + x^2 - \sqrt{1+x^2}}{(1+x^2)} > 0$

f(x) ↑ is $(-\infty, \infty)$

Q.21 (B,C)

$g(x) = 2f\left(\frac{x}{2}\right) + f(1 - x)$

$g'(x) = f'\left(\frac{x}{2}\right) - f'(1 - x) = 0 \Rightarrow \frac{x}{2} = 1 - x$

$\Rightarrow \boxed{x = \frac{2}{3}}$ critical point.

$g''(x) = \frac{1}{2} f''\left(\frac{x}{2}\right) + f''(1 - x)$

$f''(x) < 0$ for $0 \le x \le 1$

$f''(x/2) < 0$

$0 \le \frac{x}{2} \le \frac{1}{2}$

$0 \le 1 - x \le 1 \Rightarrow f''(1 - x) < 0$

$g''(x) < 0 \Rightarrow g'(x)$ is decreasing function

(4), (1) because ↓function

⊕ Θ (points 0, 2/3, 1)

$x \in \left[0, \frac{2}{3}\right]$ ↑ and $\left[\frac{2}{3}, 1\right]$ ↓

Q.22 (B,C)

$f(x) = \sin x + \cos x$

$f'(x) = \cos x - \cos x = 0$

$\tan x = 1$

$x = n\pi + \frac{\pi}{4}$

$n = 0, x = \frac{\pi}{4}$

⊕ Θ ⊕ (points 0, $\frac{\pi}{4}$, $\frac{5\pi}{4}$, 2π)

$n = 1,\ x = \frac{5x}{4}$

$\downarrow$ is $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$\uparrow$ is $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$

Q.23 (A,B,C)

$f(x) = \tan^{-1} x - \frac{1}{2}\ \ell n x$

$f'(x) = \frac{1}{1+x^2} - \frac{1}{2x} = \frac{2x-(1+x^2)}{2x(1+x^2)} = \frac{-(x-1)^2}{2x(1+x^2)}$

Θ (sign chart: 0)

f(x) is $\downarrow$ for x > 0

Greatest value will be at $x = \frac{1}{\sqrt{3}}$

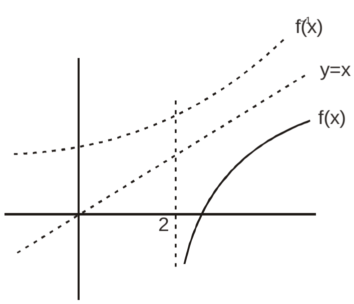
$f\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1} \frac{1}{\sqrt{3}} - \frac{1}{2}\ \ell n \left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{1}{4}\ \ell n 3$

least value will be at $x = \sqrt{3}$

$f(v) = \tan^{-1} \sqrt{3} - \frac{1}{2}\ \ell n \sqrt{3} = \frac{\pi}{3} - \frac{1}{4}\ \ell n 3$

Q.24 (A,C,D)

$f(x) = \log(x-2) - \frac{1}{x}$



$f'(x) = \frac{1}{x-2} + \frac{1}{x^2} = \frac{x^2+x-2}{x^2(x-2)}$

$f'(x) = \frac{(x+2)(x-1)}{x^2(x-2)}$

$\Theta \quad \oplus \quad \oplus \quad \Theta \quad \oplus$ (sign chart: –2, 0, 1, 2)

$f^{-1}(x)$ is monotonic increasing.

Q.25 (A, D)

(A) f(0) = 0
f(0) = 0
$f(\pi/2) = e^{\pi/2}$
Rolle's theorem Not applicable
(b) f(–1) = f (3/2) = 0
(c) f(x) = sin |x|
differentiation is $[\pi, 2\pi]$
$f(x) = 0 = f(2\pi)$

$D = \sin \frac{1}{x}$

Q.26 (A,B)
$y = 2x^2 - \ell n |x|$

$y' = 4x - \frac{1}{x} = \frac{4x^2-1}{x} = \frac{(2x-1)(2x+1)}{x}$

$\Theta \quad \oplus \quad \Theta \quad \oplus$ (sign chart: $-\frac{1}{2}$, 0, $\frac{1}{2}$)

$I_1 : x \in \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

$I_2 : x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

Q.27 (A,D)
$\phi'(x) = 3f^2 f' - 6 ff' + 4f' + 5 + 3 \cos x - 4 \sin x$
$= f'(x) [3f^2 - 6f + 4]$
$\downarrow$
d < 0
$\phi'(x)$ increasing as f increasing

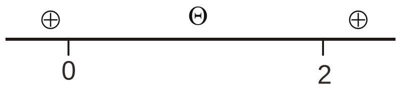
Q.28 (A,C)
$\phi(x) = f(x) + f(2a - x)$
$\phi'(x) = f'(x) - f'(2a - x)$
$x = 2a - x \Rightarrow \boxed{x = a}$ critical points
$\phi''(x) = f''(x) + f''(2a - x)$
$f''(x) > 0$ for $0 \le x \le 2a$
$f''(2a - x) > 0$ $\quad 0 \le 2a - x \le 2a$
$\phi'(x) > 0$
$\phi'(x)$ is increasing

$\Theta \quad \oplus$ (sign chart: 0, a, 2a)

$x \in (0, a) \downarrow$
$x \in (a, 2a) \uparrow$

Q.29 (A, B)
$f(x) = 3x^4 + 4x^3 - 12x^2 - 7$
$f'(x) = 12x^3 + 12x^2 - 24x$

$= 12\,x\,(x^2 + x - 2)$
$= 12\,x\,(x + 2)\,(x - 1)$



↓ is $(-2, 0) \cup (1, \infty)$
↑ $x \in (-\infty, 0) \cup (2, \infty)$

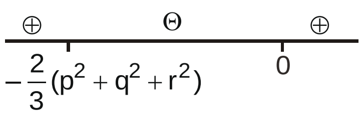
Q.30 (C,D)

$f(x) = x + 1\ \dfrac{1}{x-1}$

$f'(x) = 1 - \dfrac{1}{(x-1)^2} = 0$

x = 0, 2 are critical point
decreasing $x \in [0, 1)$ u $(1, 2]$
increasing $x \in (-\infty\ , 0)$ U $[\,2, \infty)$

Q.31 (A,B)
$f(x) = x^2\,(x + p^2 + q^2 + r^2)$
By using determinant prop.
$f'(x) = 2x\,(x + p^2 + q^2 + r^2) + x^2$
$= x\,[3x + 2p^2 + 2q^2 + 2r^2]$



↑ $x < -\dfrac{2}{3}\,(p^2 + q^2 + r^2) \cup x > 0$

↓ $x \in \left(-\dfrac{2}{3}(p^2+q^2+r^2), 0\right)$

Q.32 (A,C)
(A) $f(z) = \tan^{-1} z$

$f'(z) = \dfrac{f(y)-f(x)}{y-x}$

$\left|\dfrac{1}{1+z^2}\right| = \left|\dfrac{\tan^{-1}y - \tan^{-1}x}{y-x}\right|$

$\left|\dfrac{\tan^{-1}y - \tan^{-1}x}{y-x}\right| \le 1$

Aliter
$|\tan^{-1} x - \tan^{-1} y| \le |y - x|$

This will be is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

But RHS can infinite so always true.
(B) $f(z) = \sin z$
$f'(z) = \cos z$

$\dfrac{f(y)-f(x)}{y-x} = \cos z \Rightarrow \left|\dfrac{\sin y - \sin x}{y-x}\right| = |\cos z| \le 1$

$|\sin y - \sin x| \le |y - x|$
Aliter
$|\sin y - \sin x| \le\ |y - x|$
↓
LHS as $x \to \infty$,
a finite quantity between – 1 to 1
RbS will be infinitely
which is always two.

Q.33 (A,B,C,D)
$f'(x) = 16\,x^3\,(\,3 \ln x - 1) = 0$
$x = e^{1/3}$ is mining
$f''(x) = 16\,(9x^2 \ln x) =\ 0$
upward $(1, \infty\,)$
downward $(0, 1)$
$x = 1 \Rightarrow f(1) = -7$

Q.34 (B, D)
$x = r\cos\theta,\ y = r\sin\theta$

$\Rightarrow \dfrac{4}{r^2} = 5(1+\cos 2\theta) + 3\sin 2\theta + 1 - \cos 2\theta$

$= 6 + 4\cos 2\theta + 3\sin 2\theta$, which has maximum 11 and minimum 1

$\therefore$ OP has minimum $\dfrac{2}{\sqrt{11}}$ and maximum 2.

Q.35 (A, B, D)

$f(x) = \dfrac{x+1}{x^2+1}$

$(x^2 + 1)\,f'(x) + 2x\,f(x) = 1$

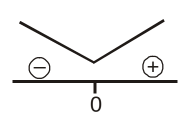
$f'(x) = \dfrac{3x^2+2x+1}{(x^2+1)^2} \ne 0$

$f''(x) = \dfrac{x^3+3x^2-3x-1}{(x^2+1)^4} = 0$

$x^3 + 3x^2 - 3x - 1 = 0$
$(x - 1)\,(x^2 + 4x + 1) = 0$
$x = 1,\ \ (x^2 + 4x+1)=0\ \ \Rightarrow x = -2 \pm \sqrt{3}$

Q.36 (D)

$f(x) = \dfrac{x^2-1}{x^2+1}$



$f'(x) = \dfrac{4x}{(x^2+1)^2}$

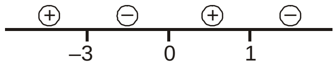
x = 0 is minima

$f(0) = -1$

Q.37 (A,C,D)
$f(x) = 40\,(3x^4 + 8x^3 - 18x^2 + 60)^{-1}$
$f'(x) = -\,40\,(3x^4 + 8x^3 - 18x^2 + 60)^{-2}$
$(12x^3 + 24x^2 - 36x)$
$f'(x) = 0$

$\Rightarrow\ 12x(x^2 + 2x - 3) = 0$

$12x\,(x + 3)\,(x - 1) = 0$



$x = -3, 1$ Local Maxima

$x = 0$ Local minima

Q.38 (B)

$f(x) = a\,\ell n\,|x| + bx^2 + x$

$f'(x) = \dfrac{a}{x} + 2bx + 1$

$f'(-1) = 0 \Rightarrow -a - 2b + 1 = 0$ .....(1)

$f'(2) = 0 \Rightarrow \dfrac{a}{2} + 4b + 1 = 0$ .....(2)

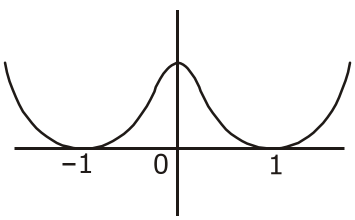
Solving (1) & (2)

we get $a = 2,\ b = -1/2$

Q.39 (A,C,D)

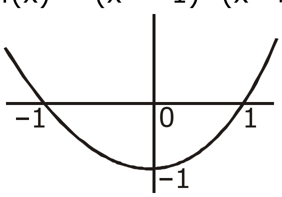
If $n = 2$

$f(x) = (x^2 - 1)^2\,(x^2 + x + 1)$



If $n = 3$

$f(x) = (x^2 - 1)^3\,(x^2 + x + 1)$



Q.40 (A,C)

$$f'(x) = \frac{3(\sin^{-1}x)^2}{\sqrt{1-x^2}} - \frac{3(\cos^{-1}x)^2}{\sqrt{1-x^2}} = 0$$

$\sin^{-1}x = \cos^{-1}x \Rightarrow x = \dfrac{\pi}{4}$

critical points are $x = 1, -1, \dfrac{\pi}{4}$

$f\left(\dfrac{\pi}{4}\right) = \dfrac{\pi^3}{32}$;

$f(-1) = -\dfrac{\pi^3}{8} + \pi^3 = \dfrac{7\pi^3}{8}$

Q.41 (B,D)

$y = f(x) = \dfrac{x}{1 + x\tan x}$

y_{max} when is reciprocal take min. value

$\dfrac{1}{y} = \dfrac{1 + x\tan x}{x} = \dfrac{1}{x} + \tan x$

Let $z = \dfrac{1}{x} + \tan x$

$\dfrac{dz}{dx} = -\dfrac{1}{x^2} + \sec^2 x = 0 \quad \Rightarrow \quad x = \cos x$

$\dfrac{d^2z}{dx^2} = \dfrac{2}{x^3} + 2\sec^2 x \tan x > 0$

min. at $x = \cos x$

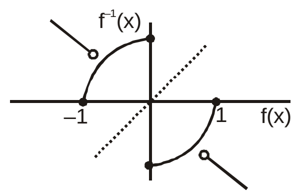
y take max. value at some x_0

where $x_0 = \cos x$

Q.42 (A,C)

$f(X) = -\sqrt{1 - x^2}\ ,\qquad 0 \le x \le 1$

$\qquad -x \qquad\qquad x > 1$



Q.43 (B,D)

$x = \phi(t) = t^5 - 5t^3 - 20t + 7$

$y = \psi(t) = 4t^3 + 4t^2 - 18t + 3$

$\dot{x} = \phi'(t) = 5t^4 - 15t^2 - 20$

$= 5\,(t^2 - 4)\,(t^2 + 1)$

$\dot{x} \ne 0$ when $-2 < t < 2$

$\dot{y} = \psi'(t) = 12t^2 + 8t - 18$

$= 6\,(2t^2 - t - 3)$

$= 6\,(2t - 3)\,(t + 1)$

$\dot{y} = 0 \Rightarrow t = 3/2,\ t = -1$

$-2 < t < 2$ satisfied

$\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = 0 \qquad (x \ne 0)$

$\dot{y} = 0 \Rightarrow t = -1, 3/2$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right).\frac{dt}{dx} = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{(\dot{x})^2}\cdot\frac{1}{(\dot{x})}$$

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\,\ddot{y}}{\dot{x}^2} \times \frac{1}{\dot{x}}$$

$$\frac{d^2y}{dx^2} = \frac{\ddot{y}}{\dot{x}^2} = \frac{6(4t-1)}{\dot{x}^2}$$

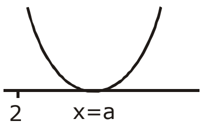
at $t = -1 \qquad \dfrac{d^2y}{dx^2} < 0$

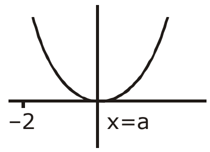
at $t = 3/2$ $\qquad \dfrac{d^2y}{dx^2} > 0$

$t = -1 \quad y_{max} = 14$

$t = 3/2 \quad y_{min} = -\dfrac{69}{4}$

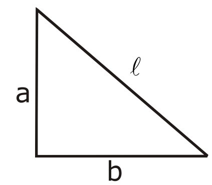
Q.44 (A,D)





Q.45 (A,C)
In all three definition separately differentiate & check.

Q.46 (A,C)



$a + b = 9$

$v = \dfrac{1}{3}\pi r^2 h$

$= \dfrac{1}{3}\pi b^2 a = \dfrac{1}{3}\pi b^2 (9 - b)$

$\dfrac{dv}{db} = 0 \Rightarrow b = 6 \Rightarrow a = 3$

$\ell = \sqrt{a^2 + b^2} = 3\sqrt{5}$

Surface area $= \pi r \ell$

$= \pi b \ell$

$= \pi (6)(3\sqrt{5}) = 18\sqrt{5}\,\pi$

Q.47 (A,B,C)

$f(x) = \sin x - x \cos x$

$f'(x) = \cos x + x \sin x - \cos x$

$f'(x) = x \sin x = 0$

$\Rightarrow \quad x = 0 \quad \sin x = 0$

(reject) $\quad x = n\pi$

$f'(x) = \sin x + x \cos x$

$x = \pi \quad f''(\pi) = -\pi < 0$ max.

$x = 2\pi \quad f''(2\pi) = 2\pi > 0$ min.

$x = -\pi \quad f''(-\pi) = \pi > 0$ min.

$x = -2\pi \quad f''(-2\pi) = -2\pi < 0$ max.

Q.48 (A,C,D)
check for

$f(1 + x) = f(1 - x)$

& $f(2 + x) = f(2 - x)$

$f''(1) < 0$

$\Rightarrow x = 1$ is point of maxima.

Q.49 (B,C)

$y = \dfrac{ax^2 + 2bx + c}{Ax^2 + 2Bx + C}$

Cross multiply

$ax^2 + 2bx + c - y(Ax^2 + 2Bx + C) = 0$

$D \geq 0 \quad \forall x \in R$

Let $y \in [\alpha, \beta]$

Equality holds when $D = 0$

$\Rightarrow \alpha, \beta$ are the extremum values

Q.50 (A, C)
From options we can check by putting the value in function instead of differentiatig the function.

Comprehension # 1 (Q. No. 51 to 53)

Q.51 (B)

Q.52 (D)

Q.53 (C)

(51 to 53)

$\dfrac{da}{dt} = 2 \qquad \Rightarrow a = 2t + c$

$\because \quad c = 0 \qquad \{\because \quad a = 0$, when $t = 0\}$

$\therefore \quad a = 2t$

$\therefore$ the curve $y = x^2 - 2ax + a^2 + a$ becomes

$y = x^2 - 4tx + 4t^2 + 2t$

if $x = 0$, then $y = 4t^2 + 2t$

$\dfrac{dy}{dx} = 2x - 4t \quad \therefore \left.\dfrac{dy}{dx}\right|_{at\ x=0} = -4t$

$\therefore$ equation of the tangent

$y - (4t^2 + 2t) = -4t(x - 0)$

i.e. $\quad y = -4tx + 4t^2 + 2t$

vertex of $y = x^2 - 4tx + 4t^2 + 2t$ is $(2t, 2t)$

$\therefore$ distance of vertex from the origin $= 2\sqrt{2}\,t$

$\therefore$ rate of change of distance of vertex from origin with respect to $t = 2\sqrt{2}$

i.e $\quad k = 2\sqrt{2}$

$c(t) = 4t^2 + 2t$

$\therefore \dfrac{dc}{dt} = 8t + 2 \quad \therefore \left.\dfrac{dc}{dt}\right|_{at\ t = 2\sqrt{2}} = 16\sqrt{2} + 2$

$\therefore \ell = 16\sqrt{2} + 2$

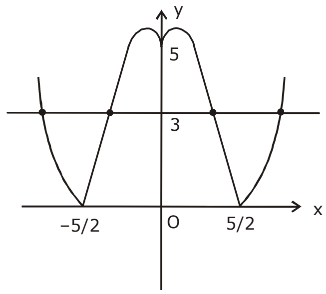
$m(t) = -4t$

$\therefore \dfrac{dm}{dt} = -4 \quad \therefore \left.\dfrac{dm}{dt}\right|_{at\ t = \ell} = -4$

Comprehension # 4 (Q. No. 54 to 56)

Q.54 (A)
Q.55 (B)
Q.56 (D)

54



55 $f'(x) = -\frac{3}{2}x^2 - \frac{3}{2}x + 3$

$f'\left(\frac{5}{2}\right) = -\frac{3}{2} \times \frac{25}{4} - \frac{3}{2} \times \frac{5}{2} + 3$

$= -\frac{75}{8} - \frac{15}{4} + 3$

$= -\frac{81}{8}$

Slope of normal = $\frac{8}{81}$

Equation of normal

$y - 0 = \frac{8}{81}\left(x - \frac{5}{2}\right)$

$\Rightarrow 81y = 8x - 20$

or $8x - 81 - 2\ 0 = 0$

56 $f'(0) = 3$

$\therefore$ Equation of tangent at Q (0, 5) is

$y - 5 = 3(x - 0)$

$3x - y + 5 = 0$

Comprehension # 2 (Q. No. 57 & 58)

Q.57 (C)

Q.58 (B)

(57 to 58)

Let $g(x) = \frac{x + \sin x}{2}$, $x \in [0, \pi]$. g(x) is increasing function of x.

$\therefore$ range of g(x) is $\left[0, \frac{\pi}{2}\right]$

$\therefore$ $f(x) = \frac{x + \sin x}{2}$, $x \in [0, \pi]$

Now let $\pi \le t \le 2\pi$, then

$f(t) + f(2\pi - t) = \pi$

i.e $f(t) + \frac{2\pi - t + \sin(2\pi - t)}{2} = \pi$

i.e $f(t) + \pi - \frac{t}{2} - \frac{\sin t}{2} = \pi$

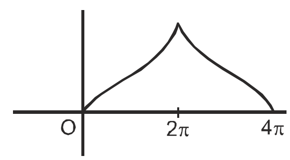
i.e $f(t) = \frac{t + \sin t}{2}$

$\therefore$ $f(x) = \frac{x + \sin x}{2}$ for $\pi \le x \le 2\pi$

Thus $f(x) = \frac{x + \sin x}{2}$ for $0 \le x \le 2\pi$

Also $f(x) = f(4\pi - x)$ for all $x \in [2\pi, 4\pi]$

$\Rightarrow$ f(x) is symmetric about $x = 2\pi$



Figure

$\therefore$ from graph of f(x)

$\therefore$ $\alpha = 2\pi - 0 = 2\pi$

$\therefore$ $\beta = \alpha$

Maximum value is $f(2\pi) = \pi = \frac{\beta}{2}$

Comprehension # 3 (Q. No. 59 to 61)

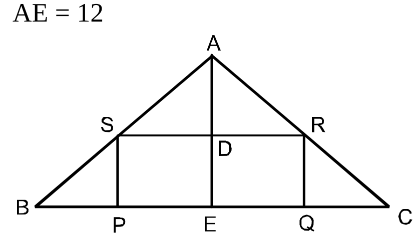
Q.59 (A)

Q.60 (B)

Q.61 (C)

(64 to 66)

AE = 12



Figure

$\Delta ABC, \Delta ASR$ are similar triangles

$\Rightarrow \frac{AD}{AE} = \frac{SR}{BC}$

$AD = \frac{12}{36} SR = \frac{SR}{3}$

Area PQRS = SR. DE

$= SR\,.\,(12 - \frac{SR}{3})$

Maximum PQRS = $\frac{0 - 12^2}{4\left(\frac{-1}{3}\right)} = 108$

Length PR = $\sqrt{SP^2 + SR^2}$

$$= \sqrt{\left(12 - \frac{SR}{3}\right)^2 + SR^2}$$

$$= \sqrt{\frac{10}{9}SR^2 - 8SR + 144}$$

Minimum PR = $\sqrt{\dfrac{4.\frac{10}{9}.144 - 8^2}{4.\frac{10}{9}}} = \sqrt{\dfrac{648}{5}}$

Comprehension # 5 (Q. No. 62 & 63)

Q.62 (C)

Q.63 (B)

62 $a^2x^3 + abx^2 + acx + ad = 0$
has three real roots.
and from Eq. (i) $ax^3 + bx^2 + cx + d = 0$
has two positive and one negative real roots.

63 $\because \int_0^1 (1 + e^{x^2})(ax^3 + bx^2 + cx + d)dx$

$\because \int_0^2 (1 + e^{x^2})(ax^3 + bx^2 + cx + d)dx = 0$

$\Rightarrow x \in [0,2]$

Q.64 (A) p; (B) r,s; (C) q, t

(A) $\because 4y\frac{dy}{dx} = 2ax$

$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{a}{-2} = -1$

$\Rightarrow a = 2$

Also (1, –1) lies on $2y^2 = ax^2 + b$

$\Rightarrow$ 2 = a + b

$\therefore$ a = 2 , then b = 0

$\Rightarrow$ a – b = 2(P)

(B) $\because$ (a, b) lies on the curve $9y^2 = x^3$

Therefore, $9b^2 = a^3$

$18y\frac{dy}{dx} = 3x^2$

and $\frac{dy}{dx} = \frac{x^2}{6y} \Rightarrow \left.\frac{dy}{dx}\right|_{(a,b)} = \frac{a^2}{6b}$

Since, the normal to hte curve at (a, b) make equal intercepts with the coordinate axes. Therefore slope of normal = ± 1

$\Rightarrow -\dfrac{1}{\left.\frac{dy}{dx}\right|_{(a,b)}} \Rightarrow \left.\frac{dy}{dx}\right|_{(a,b)} = \pm 1 \Rightarrow \frac{a^2}{6b} = \pm 1$

$\Rightarrow a^2 = \pm 6b \Rightarrow a^4 = 36b^2 = 36\left(\frac{a^3}{9}\right)$

$\therefore$ a = 0, 4 then b = 0, $\pm 8/3$

But the line making equal intercepts with the coordinate axes can not pass through the origin

Hence, the required points are (4, 8/3) and (4, —8/3)

$\therefore$ a – b = 4/3 (R) and a – b = 20 /3

and a +|b| = 4 + 8/3 = 20/3 (S)

(C) $\because$ (1, 2) lies on $y = ax^2 + bx + 7/2$

$\therefore$ 2 = a + b + 7/2

$\Rightarrow a + b + 3/2 = 0$

$\left.\frac{dy}{dx}\right|_{(a,2)} = 2a(1) + b = 2a + b$;

Given 2a + b = $-\frac{1}{2}$

Solving Eqs. (i) and (iv) , then we get

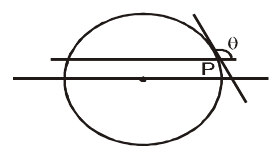
a = 1, b = –5/2

a – b = 7/2 (Q)

and 5a + 2b = 0 (T)

Q.65 (A) (p), (B) (s), (C) (q), (D) (r)

(A) $1 \le |\sin x| + |\cos x| \le \sqrt{2}$



Figure

$\Rightarrow$ y = [| sin x | + | cos x|]
y = 1
P(2, 1)
$x^2 + y^2 = 5$

$\frac{dy}{dx} = \frac{-x}{y}$

$\tan\theta = \left|\frac{-2-0}{1+0}\right|$

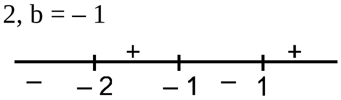
$\text{cosec}^2\theta = \frac{5}{4}$

(B) Length of subnormal = $\left|y\frac{dy}{dx}\right|$

$$= \left|-3\sin\left(\frac{-\pi}{4}\right)\frac{-3\cos\left(\frac{-\pi}{4}\right)}{\left(-\sqrt{2}\sin\left(\frac{-\pi}{4}\right)\right)}\right| = \frac{3}{\sqrt{2}}\frac{3}{\sqrt{2}} = \frac{9}{2}$$

(C) $f'(x) = 12(x + 2)(x + 1)(x - 1)$

$\Rightarrow$ a = – 2, b = – 1



sings of f'(x)

(D) $\frac{a^3+b^3}{48} = \frac{a^3+(8-a)^3}{48} = \frac{8}{48}(3a^2 - 24a + 64)$

Minimum $\frac{a^3+b^3}{48}$ is $\frac{8}{48}\frac{(4.3.64-24^2)}{4.3} = \frac{8}{3}$

Q.66 (A) (p,q); (B) (r,s); (C) (r,s); (D) (r,s)

(A) f(x) is continuous and differentiable f(0) = f(π)

Hence condition in Rolle's theorem and LMVT are satisfied.

(B) $f(1^-) = -1$, $f(1) = 0$, $f(1^+) = 1$

f(x) is not continuous at x = 1, belonging to $\left[\frac{1}{2},\frac{3}{2}\right]$

Hence, atleast one condition in LMVT and Rolle's theorem is not satisfied

(C) $f'(x) = \frac{2}{5}(x-1)^{-3/5}, x \neq 1$

At x = 1, f(x) is not differentiable.

Hence at least one condition in LMVT and Rolle's theorem is not satisfied.

(D) At x = 0

$$\text{L.H.D.} = \underset{x\to0^-}{\text{limt}} \frac{x\left(\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right)-0}{x-0} = \frac{0-1}{0+1} = -1$$

R.H.D. = 1

At x = 0, f(x) is not differentiable

Hence at least one condition in LMVT and Rolle's theorem is not satisfied.

Q.67 (A) p,s,(B) p,q,r,s,t (C) p,q

(A) $\because\ f(x) = \frac{x}{(1+x^2)}$

$\therefore\ f'(x) = \frac{(1-x^2)}{(1+x^2)^2}$

$\because\ f'(x) < 0 \Rightarrow \frac{1-x^2}{(1+x^2)^2} < 0$

$\Rightarrow (1-x^2) < 0 \Rightarrow x^2 - 1 > 0$

$\therefore\ x \in (-\infty,-1)\cup(1,\infty)(P,S)$

(B) $f(x) = \tan^{-1}x - x$

$\therefore\ f'(x) = \frac{1}{1+x^2} - 1 = -\frac{x^2}{1+x^2} < 0$

$\therefore\ f'(x) < \forall x \in R(P,Q,R,S,T)$

(C) $\because\ f(x) = x - e^x + \tan\left(\frac{2\pi}{7}\right)$

$\therefore\ f'(x) = 1 - e^x > 0$

or $e^x < 1$ or $e^x < e^0$

$\therefore\ x < 0$

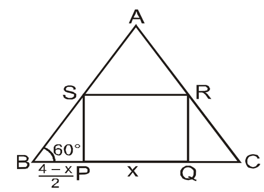
$x \in (-\infty, 0)(P,Q)$

Q.68 (A) (r), (B) (s), (C) (q), (D) (p)

(A) Let PQ = x

Then BP = $\frac{4-x}{2}$

$\therefore$ PS = $\frac{4-x}{2}\tan60^o = \frac{\sqrt{3}(4-x)}{2}$



Figure

$\therefore$ area A of rectangle = $\frac{\sqrt{3}}{2}(4-x)x$

$\frac{dA}{dx} = \frac{\sqrt{3}}{2}(4-2x) = 0 \Rightarrow x = 2$

$\frac{d^2A}{dx^2} = -\sqrt{3} < 0$

$\therefore$ A is maximum, when x = 2.

$\therefore$ Maximum area = $\frac{\sqrt{3}}{2}2.2 = 2\sqrt{3}$.;

Square of maximum area = 12

(B) Dimensions be x, 2x, h

72 = x . 2x . h

$36 = x^2h$ ....(1)

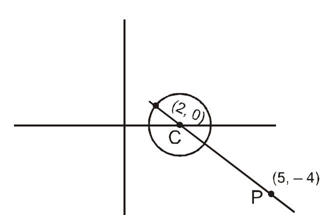
$S = 4x^2 + 6xh$

$S = 4x^2 + 6\frac{36}{x}$

$\frac{dS}{dx} = 8x - \frac{216}{x^2} = \frac{8(x^3-3^3)}{x^2}$

For least S, x = 3 and least S is 108.

(C) Let y = $\sqrt{-3+4x-x^2}$



Figure

$x^2 + y^2 - 4x + 3 = 0$

$(x-2)^2 + y^2 = 1$, center C = (2, 0)

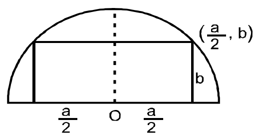
Consider point P(5, – 4)

$CP = \sqrt{9+16} = 5$

Maximum value of

$\left(\sqrt{-3+4x-x^2}+4\right)^2 + (x-5)^2$ is $(5+1)^2 = 36$.

(D)



Figure

$x^2 + y^2 = 5$

$\frac{a}{2} = \sqrt{5}\cos\theta$, $b = \sqrt{5}\sin\theta$

Let $f(\theta)$ be perimeter

$f(\theta) = 2a + 2b$

$= 2\sqrt{5}(2\cos\theta + \sin\theta)$

$f'(\theta) = 2\sqrt{5}(-2\sin\theta + \cos\theta)$

$f''(\theta) = 2\sqrt{5}(-2\cos\theta - \sin\theta)$

$f'(\theta) = 0 \Rightarrow \tan\theta = \frac{1}{2}$ and $f''(\theta) < 0$

$\Rightarrow$ $f(\theta)$ is greatest

a = 4, b = 1

$a^3 + b^3 = 65$

Q.69 (A) r,t (B) q (C) p,s

(A) The function f(x) is differentiable except at x = 0

$\therefore$ f(x) is not continuous at x = 0, also

$\because f(x) = 2x^2 + \frac{2}{x^2} \Rightarrow f'(x) = 4x - \frac{4}{x^3}$

Critical points are x =1, –1

Now , values of f(x) at x = –2, –1, 0, 1, 2

are f(–2), f(–1), f(0),f(1), f(2)

i.e. $\frac{17}{2}, 4, 1, 4, \frac{17}{2} \Rightarrow G = \frac{17}{2}, L = 1 \Rightarrow [G + L]$

= 9, (G + L) = 10 (R, T)

(B) Given $f(x) = x^3 - 6x^2 + 9x + 1$

$\therefore f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$

$\therefore x = 1, 3$ but $3 \in [0, 2]$

G = max {f(0), f(1), f(2)} = max {1, 5, 3} = 5

L=min{f(0), f(1), f(2)} = min {1,5,3} =1

$\Rightarrow$ G = 5, L =1 $\Rightarrow$ [G + L] = 6, (G + L) = 6(Q)

(C) Given f(x) = arc tan $x - \frac{1}{2}\ln x$

NUMERICAL VALUE BASED

Q.1 [4]

$$x^4 + y^4 = a^4$$

$$4x^3 + 4y^3\frac{dy}{dx} = 0 \quad \frac{dy}{dx} = \frac{-x_1^3}{y_1^3} \quad y - y_1 = \frac{-x_1^3}{y_1^3}(x - x_1) \quad y = 0 \quad x = \frac{a^4}{x_1^3}$$

$$y - y_1 : \frac{x_1^4}{y_1^3} y = \frac{a^4}{y_1^3}$$

$$P = \frac{a^4}{x_1^3} \quad P^{-4/3} = \frac{a^{-16/3}}{x_1^{-4}}$$

$$q^{-4/3} = \frac{a^{-16/3}}{y_1^{-4}}$$

$$p^{-4/3} + q^{-4/3} = a^{-16/3}.a^4$$

$a^{-4/3}$ k = 4

Q.2 [3]

y = 3x + 3 but $f'(0) = -\frac{1}{3}$

$$\lim_{x \to 0} \frac{2x}{2xf'(x^2) - 40^x f'(4x^2) + 56^6 f'(7x^2)}$$

$$\frac{1}{f'(0) - 20f'(0) + 28f'(0)}$$

$$\frac{1}{9f'(0)} = -\frac{1}{9\left(-\frac{1}{3}\right)} \Rightarrow k = 3$$

Q.3 [6]

Let tangent is at P (h, k)

$2xy + x^2\frac{dy}{dx} = 0$ is $\frac{dy}{dx} = \frac{-2y}{x} = \frac{-2k}{h}$

Eq^n of tangent $y - K = \frac{-2K}{h}(x - h)$

X–intercept $\Rightarrow$ $a = \frac{3h}{2}$,

Y–intercept $\Rightarrow$ 3k = b

$$\frac{a^2b}{c^3} = \frac{9h^2.3k}{4c^3} = \frac{27.h^2k}{4c^3} = \frac{27}{4}$$

$\left(\text{as } h^2K = C^3\right)$

$$[\mu]=\left[\frac{27}{4}\right]=7$$

Q.4 [4]

$$x^2+y^2=a^2\sqrt{2}\ \ and\ \ x^2-y^2=a^2$$

$$\frac{dy}{dx}=\frac{-x}{y};\ \frac{dy}{dx}=\frac{x}{y}$$

$$x^2=\frac{a^2\left(\sqrt{2}+1\right)}{2}\ y^2=\frac{a^2\left(\sqrt{2}-1\right)}{2}$$

$$\tan^{-1}\left(\frac{x}{y}+\frac{x}{y}\right)/1-\frac{x^2}{y^2}=\ \tan^{-1}\frac{2\left(\sqrt{2}+1\right)}{1-\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)}=\tan^{-1}\frac{2(2-1)}{-2}$$

$$\left|\tan^{-1}(-1)\right|=\frac{\pi}{4}$$

Q.5 [3]

$$Let\ \ f(x)=\sqrt{x-1}+\sqrt{2-x}\ \ \text{g(x) = x}^2\text{ – bx + c}$$

$$f'(x)=\frac{1}{2\sqrt{x-1}}\frac{-1}{2\sqrt{2-x}}=0$$

$$g'(x)=2x-b$$

$$\sqrt{2-x}=\sqrt{x-1}\qquad g'(x)=0\ \ \text{b = + 3}$$

$$\Rightarrow x=\frac{3}{2}\ \ f'(x)=0$$

Q.6 [1]

$f(x)=x/\log_e x$ has domain $(0,\infty)-\{1\}$

$f(x)$ is decreasing when $f^1(x)<0$

$$\frac{\log_e x1-x\frac{1}{x}}{(\log_e x)^2}<0\Rightarrow\ \ x<e$$

$\therefore f(x)$ is decreasing on $(0,1)$ U $(1,e)$

$\Rightarrow$ no of integer value of x is 1

Q.7 [1]

Given $f:[1,10]\Rightarrow[1,10]$

is a non decreasing function

$g:[1,10]\to[1,10]$ is a non increasing function.

$h(x)=fog(x):[1,10]\to[1,10]$ is a decreasing function.

$\therefore h(1)\ge 1\to 1$

But $x\ge 1\Rightarrow\ h(x)\le h(1)\Rightarrow\ h(x)\le 1\to 2$

From 1, 2, $h(x)=1\ \ \forall x\in[1,10]$

So $h(7)=1$.

Q.8 [2]

Given that x and y are two real variables such that x>0 and xy=1

To find the minimum value of x+y

Let $S=x+y$

$\Rightarrow S=x+\frac{1}{x}$ (using xy=1)

For minimum value of S, differentiating of above with respect to x,we get

$$\frac{dS}{dx}=1-\frac{1}{x^2}$$

For minimum value of $S,\frac{dS}{dx}=0$

$$\Rightarrow 1-\frac{1}{x^2}=0\ \ \Rightarrow x^2=1\Rightarrow x=\pm 1$$

But $x>0,\therefore x=1$

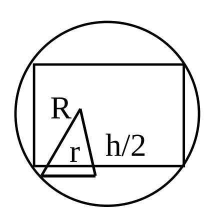
Now $\frac{d^2S}{dx^2}=\frac{2}{x^3}\ \ \Rightarrow \left.\frac{d^2S}{dx^2}\right|_{x=1}=2=+ve$

$\therefore$ S is minimum when x=1

$$\therefore S_{\min}=1+\frac{1}{1}=2$$

Q.9 [3]

$$V=2\pi r^2\sqrt{R^2-r^2}$$



$$V'(r)=0 \ we \ \ get \ \sqrt{\frac{2}{3}}R=r\Rightarrow h=8\sqrt{3} \ \ k=3$$

Q.10 [3]

$$f(x)=x^3-3x+1$$

$$f'(x)=3(x^2-1)=0$$

$f(-1)\ f(1) < 0$

KVPY

PREVIOUS YEAR'S

Q.1 (B)

Let initially 2 bases have radii 5 cm and r cm.

Finally base have radii (1.21 × 5) and r

Ratios of volumes $=\frac{V_2}{V_1}=1.21$

$$V_2=\frac{\pi h}{3}[(6.05)^2+(6.05)r+r^2]$$

$$V_1=\frac{\pi h}{3}[5^2+5r+r^2]$$

$$\frac{V_2}{V_1}=1.21\Rightarrow\frac{(6.05)^2+(6.05)r+r^2}{5^2+5r+r^2}=1.21$$

$$\Rightarrow r^2=\frac{6.3525}{21}$$

$$\Rightarrow r=\frac{11}{2}cm=55mm$$

Q.2 (D)

Speed of B = V km/hr

Speed of A = 3V km/hr

Given 4V = 2 × 60 km/hr $\quad\Rightarrow V=30\ km/hr$

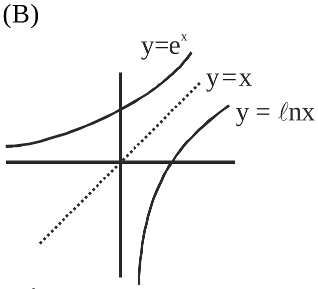
Distance covered by then after 10 min = 2 × 10 = 20 km

so remaining distance = (30 – 20) km = 10 km

Time taken by B to cover 10 km $=\frac{10}{30/60}=20\,min.$

Total time = 20 + 10 = 30 min

Q.3 (B)



II^{nd} curve

$$y'=\frac{1}{x}$$

x = ⇒ point (1, 0)

similarly Ist ⇒ point (0, 1)

distance = $\sqrt{2}$

Q.4 (D)

Here weight will be measured in two criteria's weight of bucket w + weight of water ($\pi r^2 h$ x density) in first case

$$w+\pi r^2\frac{h}{2}\rho=10$$

in second case

$$w+\pi r^2\frac{2}{3}h\rho=11$$

$$\pi r^2\left(\frac{2}{3}-\frac{1}{2}\right)h\rho=1$$

$$\pi r^2\frac{h}{6}\rho=1$$

$\pi r^2 hp = 6$

Also

$$w+\frac{\pi r^2 h\rho}{2}=10$$

w + 3 = 10

w = 7

Hence total weight is

$$w+\pi r^2 h\rho$$

= 7 + 6 =13

Q.5 (D)

$$f(x)=\frac{\sin(x-a)+\sin(x+a)}{\cos(x-a)-\cos(x+a)}=\frac{2\sin(x).\cos a}{2\sin x.\sin a}=\cot a$$

Q.6 (C)

$f(x) = x^{2n+1} - (2n + 1) x + a$

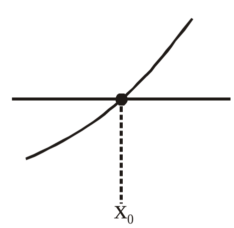
$f'(x) = (2n + 1) x^{2n} - (2n + 1)$

$= (2n + 1) (x^{2n} - 1) \le 0$ when $x \in [-1, 1]$

f(x) is strictly decreasing in [–1, 1]

f(x) cut x axis at most oen point in given interval

Q.7 (A)



$$f(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$$

$$f'(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}$$

$f''(x) = 1 + x + \frac{x^2}{2} > 0$

$\Rightarrow f'(x)$ is an increasing f^n

$\Rightarrow f'(x) = 0$ at $x = x_0$

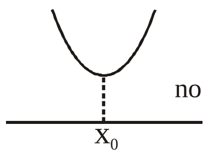
$\Rightarrow f'(x_0) = 0 \Rightarrow 1 + x_0 + \frac{x_0^2}{2} + \frac{x_0^3}{6} = 0$

....(A)

$f'(-2)\, f'(-1) < 0$

$\Rightarrow x_0 \in (-2,-1)$

$f(x_0) = 1 + x_0 + \frac{x_0^2}{2} + \frac{x_0^3}{6} + \frac{x_0^4}{24} = \frac{x_0^4}{24} > 0$



no solution

Q.8 (D)

Given that $2r + r\theta = P$ $\qquad r = \frac{P}{2+\theta}$

area $= \frac{1}{2}r^2\theta = \frac{1}{2}\theta\left(\frac{P}{2+\theta}\right)^2 = \frac{1}{2}\frac{P^2\theta}{(2+\theta)^2}$

$\frac{dA}{d\theta} = 0 \qquad \theta = 2^C$

Q.9 (D)

$f(x) = \alpha x^2 - 2 + \frac{1}{x}$

$f(x) = \frac{\alpha x^3 - 2x + 1}{x} \quad \forall x\ (0,\infty)$

so $\phi(x) = \alpha x^3 - 2x + 1$ should be positive

$\phi(x) = \alpha x^3 - 2x + 1$

$\phi'(x) = 3\alpha x^2 - 2 = 0$

$x = \pm\sqrt{\frac{2}{3\alpha}}$ point of minima

$\phi\left(\sqrt{\frac{2}{3\alpha}}\right) \geq 0$

$\sqrt{\frac{2}{3\alpha}}\left\{\alpha \cdot \frac{2}{3\alpha} - 2\right\} + 1 \geq 0$

$\sqrt{\frac{2}{3\alpha}}\left(-\frac{4}{3}\right) + 1 \geq 0$

$\sqrt{\frac{2}{3\alpha}}\left(\frac{4}{3}\right) \leq 1$

$\sqrt{\frac{2}{3\alpha}} \leq \frac{3}{4}$

$\frac{2}{3\alpha} \leq \frac{3^2}{4^2}$

$\alpha \geq \frac{32}{27}$

Q.10 (A)

$f'(x) \leq 2f(x)$

$f'(x)e^{-2x} \leq 2f(x)e^{-2x}$

$\frac{d}{dx}\left(f(x)e^{-2n}\right) \leq 0$

$g(x) = f(x)e^{-2x}$ is non-Increasing function

$x \geq 0$

$g(x) \leq g(0)$

$f(x)e^{-2x} \leq f(0)e^{-0}$

$f(x)e^{-2x} \leq 0$

$f(x) \leq 0$ $\quad$ but given f (x) is not negative

$\therefore$ f (x) = 0 $\quad$ Constant function

Q.11 (C)

$P'(x) = 12x^2 - 3 = 3(4x^2 - 1)$

In $\left(-\frac{1}{2}, \frac{1}{2}\right)$ $P'(x) < 0$

$\Rightarrow$ P (x) is decreasing $\quad \Rightarrow$ Range $\in$ (P(–1),P(A))

Range $\in (-1,1)$

Q.12 (B)

$\lim_{n\to 0} f_n(x) = f(f(f(.......\infty \text{ times}(x))$

Now for $x_1 \in \left(0, \frac{1}{2}\right)$

$f(x_1) > x_1$ as f (x) is concave - downward

Thus $f_n \to \frac{1}{2}$ as $n \to \infty$

Similarly for $x_1 \in \left(\frac{1}{2}, 1\right)$

$f(x_1) < x_1$ as f (x) is convave upward

Thus $f_n \to \frac{1}{2}$ as $n \to \infty$

Q.13 (A)

$f'(x) = 3x^2 - 6ax + 27a^2 + 9$

$= 3[x^2 - 2ax + 9a^2 + 3] = 3((x-a)^2 + 8a^2 + 3)$

$\therefore$ f '(x) is + ve for $x \in R$ so f(x) is monotonic$\uparrow$ for $x \in R$.

Q.14 (C)

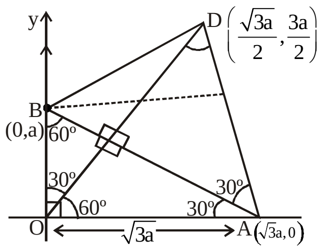
(0,3) lies on the curve

So q = 3

$$\text{Now} \frac{dy}{dx} = 2x + p; \left(\frac{dy}{dx}\right)_{(0,3)} = p = -1$$

$\therefore$ p + q = – 1 + 3 = 2

Q.15 (D)



$$M_{BD} = \frac{1}{\sqrt{3}}$$

Q.16 (C)

$P(\sin^2 x) = P(\cos^2 x)$

$P(\sin^2 x) = P(1 - \sin^2 x)$

$P(x) = P(1 - x) \ \forall \ x \in [0, 1]$

Differentiable both sides w.r.t x

$P'(x) = -P'(1 - x)$

So $P'(x)$ is symmetric about point $x = \frac{1}{2}$

So $P'(x)$ has highest degree odd

$\Rightarrow$ P(x) has highest degree even

Q.17 (A)

$S = 2\pi R^2 + 2\pi Rh + \pi R^2$

(R = radius of hemisphere & cylinder)

$$V = \frac{2}{3}\pi R^3 + \pi R^2 h$$

$$V = \frac{2}{3}\pi R^3 + \pi R^2 \times \left(\frac{5 - 3pR^2}{2\pi R}\right)$$

$$\frac{dV}{dR} = 2\pi R^2 + \frac{5}{2} - \frac{9\pi}{2} R^2$$

For maximum and minimum $\frac{dV}{dR} = 0$

$5\pi R^2 = S$

$5\pi R^2 = 3\pi R^2 + 2\pi Rh$

$2R = 2h$

$h : R = 1 : 1$

Q.18 (A)

$x_0^2 + y_0^2 > 1$ $\quad\quad$ $x_0 - y_0$ fixed

x,y arbitrary

$x^2 + y^2 \le 1$, Let $x = \cos\theta$

$y = \sin\theta$

for men

$z = (x - x_0)^2 + (y - y_0)^2$

$z = x^2 + x_0^2 + y^2 + y_0^2 - 2(x x_0 + y y_0)$

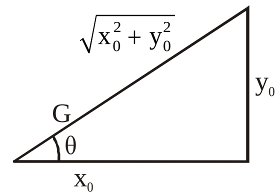
put $x = \cos\theta$, $y = \sin\theta$

$$z = x_0^2 + y_0^2 - 2\left(x_0 \cos\theta + y_0 \sin\theta\right)$$

$$\frac{dz}{d\theta} \Rightarrow 0 - 2\left(-x_0 \sin\theta + y_0 \cos\theta\right)$$

$$\frac{dz}{dz} = 0$$

$-x_0 \sin\theta = -y_0 \cos\theta$



$$\tan\theta = \frac{y_0}{x_0}$$

$$\sin\theta = \frac{y_0}{\sqrt{x_0^2 + y_0^2}}$$

$$\cos\theta = \frac{x_0}{\sqrt{x_0^2 + y_0^2}}$$

$$x = \frac{x_0}{\sqrt{x_0^2 + y_0^2}}, y = \frac{y_0}{\sqrt{x_0^2 + y_0^2}}$$

$$z = \left(\frac{x_0}{\sqrt{x_0^2 + y_0^2}} - x_0\right)^2 + \left(\frac{y_0}{\sqrt{x_0^2 + y_0^2}} - y_0\right)^2$$

$$x_0^2 \left(\frac{1}{\sqrt{x_0^2 + y_0^2}} - 1\right)^2 + y_0^2 \left(\frac{1}{\sqrt{x_0^2 + y_0^2}} - 1\right)^2$$

$$\left(x_0^2 + y_0^2\right) \frac{\left(1 - \sqrt{x_0^2 + y_0^2}\right)}{\left(\sqrt{x_0^2 + y_0^2}\right)^2}$$

$$\left(1 - \sqrt{x_0^2 + y_0^2}\right)^2 \Rightarrow \left(\sqrt{x_0^2 + y_0^2} - 1\right)^2$$

Q.19 (B)

Let Height of cone = h

Radius of base = r

And slant height = ℓ ; = $\ell = \sqrt{r^2+h^2}$

Given volume = surface area

$$\Rightarrow \frac{1}{3}\pi r^2 h = \pi r\ell + \pi r^2$$

$$\Rightarrow rh = 3\ell + 3r \Rightarrow \ell = \frac{1}{3}(rh - 3r)$$

$$\Rightarrow \sqrt{r^2+h^2} = \frac{r}{3}(h-3)$$

$$\Rightarrow r^2 + h^2 = \frac{r^2}{9}(h^2 - 6h + 9)$$

$$\Rightarrow h^2 = \frac{r^2h^2}{9} - \frac{2hr^2}{3}$$

$$\Rightarrow h = \frac{6r^2}{r^2-9} = 6 + \frac{54}{r^2-9}$$

Since h and r must be integers, and $r^2 - g$ must be a factor of 54

$r^2 - 9$ must be divisible by 3

$\Rightarrow r = 3k$

$$h = 6 + \frac{54}{9k^2-9} = 6 + \frac{6}{k^2-1}$$

Since O < $k^2 - 1$ < 6

$\Rightarrow$ k = 2 is only such value; for which h is integer

So, r = 2 × 3 = 6

$$h = 6 + \frac{6}{3} = 8$$

$\ell = 10$

is the only possible values for r and h.

Q.20 (C)

Let height of radius of cylinder are h & r respectively.

Then volume $V_1 = \pi r^2 h + \frac{2}{3}\pi r^3$ ....(A)

When height of cylinder is doubled then volume

$V_2 = 2\pi r^2 h + \frac{2}{3}\pi r^3$ ....(B)

Given that $\frac{V_2}{V_1} = \frac{3}{2} \Rightarrow \frac{2h + \frac{2}{3}r}{h + \frac{2}{3}r} = \frac{3}{2}$

$$\Rightarrow 2h + \frac{2}{3}r = \frac{3}{2}h + r$$

$\Rightarrow \frac{h}{2} = \frac{r}{3} \Rightarrow h = \frac{2}{3}r$ ......(C)

When radius is doubled then volume

$V_2^1 = 4\pi r^2 h + \frac{16}{3}\pi r^3$]

$$\frac{V_2^1}{V_1} = \frac{4h + \frac{16}{3}r}{h + \frac{2}{3}r}$$

$$\text{By}(3)\frac{V_2^1}{V_1} = \frac{4h+8h}{h+h} = 6$$

Hence volume will be increased by 500%

Q.21 (B)

$y = x^2 - 4$ & $2y = 4 - x^2$

P $(\alpha, \beta) \Rightarrow 2\beta = 4 - \alpha^2$

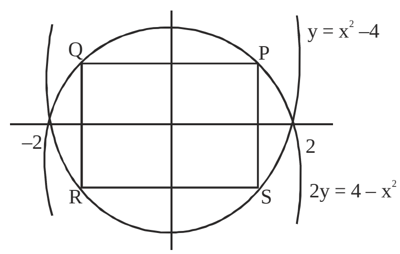
$$P\left(\alpha, \frac{4-\alpha^2}{2}\right)$$

$$Q\left(-\alpha, \frac{4-\alpha^2}{2}\right)$$

PQ = 2α

S $(\alpha, \alpha^2 - 4)$, R $(-\alpha, -4 + \alpha^2)$

$$PS = \left(\alpha^2 - 4 - \frac{4-\alpha^2}{2}\right) = \frac{3\alpha^2 - 12}{2}$$



Area = PQ PS

$$= 2\alpha. = \frac{(3\alpha^2 - 12)}{2} = 3\alpha^3 - 12\alpha = A(\alpha)$$

$$A'(\alpha) = 9\alpha^2 - 12 = 0 \Rightarrow \alpha^2 = \frac{12}{9} = \frac{4}{3}$$

$\alpha = \pm\frac{2}{\sqrt{3}}$ maximum at $\alpha = -\frac{2}{\sqrt{3}}$

Maix.=

$$A\left(-\frac{2}{\sqrt{3}}\right) = -3.\frac{8}{3\sqrt{3}} + 12.\frac{2}{\sqrt{3}} = \frac{16}{\sqrt{3}} = \frac{16\sqrt{3}}{3} = 9.22 \approx 9$$

Q.22 (A)

Assume g (x) = p(x) – x^2

(g(x) is polynomial→ differentiable function)

given p(x) > $x^2 \Rightarrow p(x) - x^2 > 0 \ \forall x \neq 0$

$\Rightarrow g(x) > 0 \quad \forall x \neq 0$

and g(0) = p(0) – 0 = 0

as g (x) > $\forall x \neq 0$

$\Rightarrow$ x = 0 should be a minima

$\therefore$ g" (x) should be ≥ 0 at x = 0

Now g'(x) = p'(x) –2x

and g''(x) = p" (x) –2

$= \frac{1}{2} - 2$

$= -\frac{3}{2}$ so, contradiction

$\therefore$ No such polynomial exist

Q.23 (B)

P(1 + x) = P(1 – x) $\Rightarrow$ P(1 + x) = – P(1 – x)

$\Rightarrow$ P'(1) = – P'(1) $\Rightarrow$ P'(1) = 0

Also P(1) = 0

So if P(x) is quadratic then P(x) = $a(x - 1)^2$

$\Rightarrow$ m = 2

Q.24 (A)

Let $g(x) = f(x)e^{-\lambda x}$; $x \in [0, 2\pi]$

so, g(0) = g(2π) = 0 (as $f(0) = f(2\pi) = 0$)

Thus, $\exists$ c $\in$ (0, 2π)

such that g'(c) = 0

$\Rightarrow f'(c) = \lambda f(c) \quad \forall \lambda \in \mathbb{R}$

$\Rightarrow S = \mathbb{R}$

Q.25 **(B)**

$f'(x) = e^x + 1 + \ell nx > 0$

(as x $\in$ [1, 2])

$\Rightarrow f(x)$ increases in [1, 2]

$\Rightarrow f_{max} = f(2) = e^2 + 2\ell n2$

Q.26 **(C)**

$x^3 + ax^2 + bx + c = 0 = (x - a)(x - b)(x - c)$

a + b + c = –a

$\Rightarrow$ 2a + b + c = 0 ...(i)

ab + bc + ca = b ...(ii)

abc = –c $\Rightarrow$ ab = –1 [$\because$ c ≠ 0] ...(iii)

Also a is a root of equation

$\Rightarrow 2a^3 + ab + c = 0 \Rightarrow 2a^3 - 1 + c = 0$

$\Rightarrow c = 1 - 2a^3$

from (1)

$2a^2 + ab + ac = 0$

$2a^2 - 1 + a(1 - 2a^3) = 0$

$2a^2 - 2a^4 + a - 1 = 0$

$2a^2(1 - a)(1 + a) + (a - 1) = 0$

$\Rightarrow (1 - a)[2a^2(a + 1) - 1] = 0$

$\Rightarrow$ a = 1 or $2a^3 + 2a^2 - 1 = 0$

when a = 1, b = $\frac{-1}{a}$ = –1 and c = $1 - 2a^3$ = –1

when $2a^3 + 2a^2 - 1 = 0$

There will be only one real solution of

.(x) = $2x^3 + 2x^2 - 1 = 0$

as $f'(x) = 6x^2 + 4x = 0 \Rightarrow x = 0, \frac{-2}{3}$

$f(0).f\left(\frac{-2}{3}\right) < 0$

$\therefore$ corresponding to this real value of a one triplet is possible

$\therefore$ Exactly two triplets (a, b, c) are possible

Q.27 **(A)**

$f(x) = \sin x + (x^3 - 3x^2 + 4x - 2)\cos x$,

x $\in$ (0,1)

$f(0)$ = –2 > 0

$f(1)$ = sin1 < 0

$\because$ f(0). $f(1) < 0 \Rightarrow f(x)$ has a zero in (0, 1)

Now,

$f(x) = \sin x + [(x - 1)^3 + (x - 1)]\cos x$

$\Rightarrow f'(x) = (3(x - 1)^2 + 2)\cos x \,.\, \sin x[(x - 1)^3 + (x - 1)]$

$= [3(x - 1)^2 + 2]\cos x + [(1 - x)^3 + (1 - x)]\sin x > 0 \ \forall x \in (0, 1)$

$\Rightarrow$ f(x) is monotone in (0, 1)

Q.28 **(C)**

f(θ) = sin(cosθ)

g(θ) = cos(sinθ)

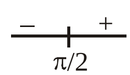
f'(θ) = cos (cosθ) (– sinθ) < 0 ∀ θ ∈ [0, π]

$\therefore$ f(θ) decreases monotonically

$\therefore$ a = max f(θ) = f(0) = sin1

b = min f(θ) = f(π) = –sin1

g'(θ) = – sin(sinθ) cosθ



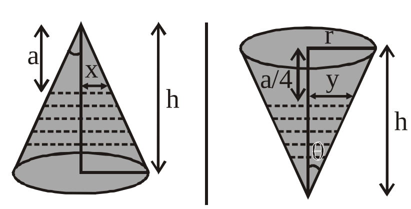
g(θ) = 1; g(π) = 1; $g\left(\frac{\pi}{2}\right) = \cos 1$

$\therefore$ c = max g (θ) = 1

d = min g(θ) = cos1

$\therefore$ b < d < a < c

Q.29 **(B)**



$\frac{r}{h} = \frac{x}{a} \Rightarrow x = \frac{ra}{h}$ $\frac{r}{h} = \frac{y}{h - \frac{a}{4}}$

$y = \frac{r}{h}\left(h - \frac{a}{4}\right)$

Equating value ofwater in both cases

$\frac{1}{3}(\pi r^2 h - \pi x^2 a) = \frac{1}{3}\pi y^2\left(h - \frac{a}{4}\right)$

$\Rightarrow r^2h - r^2h - \frac{r^2a^2}{h}.a = \frac{r^2}{h^2}\left(h - \frac{a}{4}\right)^2\left(h - \frac{a}{4}\right)$

$\Rightarrow \frac{h^2}{a^2} - \frac{h}{4a} - \frac{21}{16} = 0$

$\frac{h}{a} = \frac{\frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{21}{4}}}{2}$

$\frac{h}{a} = \frac{1 + \sqrt{85}}{8}$

JEE-MAIN
PREVIOUS YEAR'S

Q.1 (1)

$\frac{x^2}{a} + \frac{y^2}{b} = 1$ .....(1)

diff : $\frac{2x}{a} + \frac{2y}{b}\frac{dy}{dx} = 0 \Rightarrow \frac{y}{b} + \frac{dy}{dx} = \frac{-x}{a}$

$\frac{dy}{dx} = \frac{-bx}{ay}$ .....(2)

$\frac{x^2}{c} + \frac{y^2}{d} = 1$ .....(3)

diff. $\frac{dy}{dx} = \frac{-dx}{cy}$ .....(4)

$m_1m_2 = -1 \Rightarrow \frac{-bx}{ay} \times \frac{-bx}{cy} = -1$

$\Rightarrow bdx^2 = -acy^2$ .....(5)

(1) – (3) $\Rightarrow \left(\frac{1}{a} - \frac{1}{c}\right)x^2 + \left(\frac{1}{b} - \frac{1}{d}\right)y^2 = 0$

$\Rightarrow \frac{c-a}{ac}x^2 + \frac{d-b}{bd} \times \left(\frac{-bd}{ac}\right)x^2 = 0$ (Using 5)

$\Rightarrow (c - a) - (d - b) = 0$

$\Rightarrow c - a = d - b$

$\Rightarrow c - d = a - b$

Q.2 (1)

$f(x) = x^6 + ax^5 + bx^4 + x^3$

$\therefore f'(x) = 6x^5 + 5ax^4 + 4bx^3 + 3x^2$

Roots 1 & –1

$\therefore$ $6 + 5a + 4b + 3 = 0$ & $-6 + 5a - 4b + 3 = 0$

solving $a = -\frac{3}{5}$ $b = -\frac{3}{2}$

$\therefore$ $f(x) = x^6 - \frac{3}{5}x^5 - \frac{3}{2}x^4 + x^3$

$\therefore$ $5.f(2) = 5\left[64 - \frac{96}{5} - 24 + 8\right] = 144$

Q.3 (2)

$f(1) = f(2)$

$\Rightarrow 1 - a + b + 1 = 8 - 4a + 2b + 1$

$3a - b = 7$ .....(1)

$f'(x) = 3x^2 - 2ax + b$

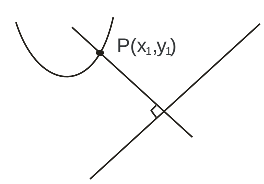
$\Rightarrow f'\left(\frac{4}{3}\right) = 0 \Rightarrow$ $3 \times \frac{16}{9} - \frac{8}{3}a + b = 0$

$\Rightarrow -8a + 3b = -16$ .....(2)

$a = 5, b = 8$

Q.4 (1)

$\left.\frac{dy}{dx}\right|_P = 1$



$\therefore$ $x_1 = 1$

$\Rightarrow P = \left(1, \frac{1}{2}\right)$

$\therefore d_{min} = \left|\frac{1 - 1 - \frac{1}{2}}{\sqrt{2}}\right| = \frac{1}{2\sqrt{2}}$

Q.5 (1)

$2 = a + b + c$ .....(i)

$\frac{dy}{dx} = 2ax + b \Rightarrow \left.\frac{dy}{dx}\right|_{(0,0)} = 1$

$\Rightarrow b = 1 \Rightarrow a + c = 1$

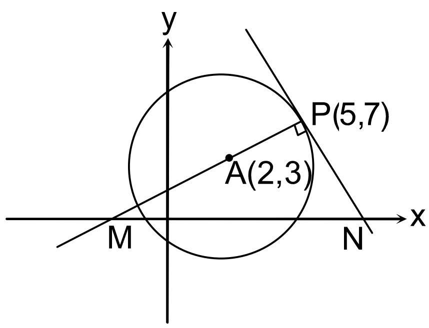
Q.6 (1)

tangent to $x^2 + 9y^2 = a$ at point $\left(\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$ is $x\left(\frac{3\sqrt{3}}{2}\right)$

$+9y\left(\frac{1}{2}\right) = 9$

$\Rightarrow$ option (1) is true

Q.7 [1225]



equation of normal at P

$(y - 7) = \left(\frac{7-3}{5-2}\right)(x - 5)$

$3y - 21 = 4x - 20$

$\Rightarrow 4x - 3y + 1 = 0$ .....(i)

$\Rightarrow M\left(-\frac{1}{4}, 0\right)$

equation of tangent at P

$(y - 7) = -\frac{3}{4}(x - 5)$

$4y - 28 = -3x + 15$

$\Rightarrow 3x + 4y = 43$ .....(ii)

$\Rightarrow N\left(\frac{43}{3}, 0\right)$

hence ar(ΔPMN) $= \frac{1}{2} \times MN \times 7$

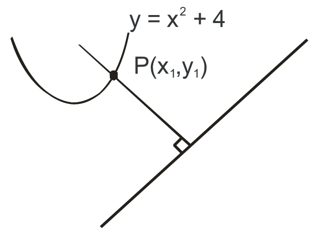
$l = \frac{1}{2} \times 175 \times 7$

$\Rightarrow 24\lambda = 1225$

Q.8 (1)

$\left.\frac{dy}{dx}\right|_p = 4$

$\therefore\ 2x_1 = 4$



$\Rightarrow x_1 = 2$

$\therefore$ Point will be (2,8)

Q.9 (1)

$$f'(x) = \begin{cases} -55 & ; \quad x < -5 \\ 6(x^3 - x - 20) & ; \quad -5 < x < 4 \\ 6(x^2 - x - 6) & ; \quad x > 4 \end{cases}$$

$$f'(x) = \begin{cases} -55 & ; \quad x < -5 \\ 6(x - 5)(x + 4) & ; \quad -5 < x < 4 \\ 6(x - 3)(x + 2) & ; \quad x > 4 \end{cases}$$

Hence, f(x) is monotonically increasing is $(-5, -4) \cup (4, \infty)$

Q.10 (2)

$m = \frac{dy}{dx} = 2x^3 - 15x^2 + 36x$

$\frac{dy}{dx} = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6) = 0$

$\Rightarrow x = 2, 3$

$\frac{d^2y}{dx^2} = 6(2x - 5)$

$\left.\frac{d^2y}{dx^2}\right|_{n=2} = -ve$

$\therefore$ Maximum at x = 2

Point (2, 46)

Q.11 [9]

Let the equation of normal is $Y - y = -\frac{1}{m}(X - x)$

Satisfy (a,b) in it $b - y = -\frac{1}{m}(1 - x)$

$\Rightarrow (b - y)dy = (x - a)dx$

by $-\frac{y^2}{2} = \frac{x^2}{2} - ax + c$ ...(i)

It passes through (3, –3) & $(4, -2\sqrt{2})$

$\therefore\ -3b-\frac{9}{2}=\frac{9}{2}-3a+c$

$\Rightarrow$ –6b–9=9–6a+2c

$\Rightarrow$ 6a–6b–2c=18

$\Rightarrow$ 3a–3b–c=9 .... (ii)

Also

$-2\sqrt{2}b-4=8-4a+c$

$4a-2\sqrt{2}\ b-c=12$ ...(iii)

Also $a-2\sqrt{2}\ b=3$ ....(iv) (given)

(ii)–(iii)$\Rightarrow -a+\left(2\sqrt{2}-3\right)b=-3$ (v)

(iv)+(v) $\Rightarrow$ b=0 a=3

= $a^2+b^2+ab=9$

Q.12 (3)

equation of tangent at $P(t,t^3)$

$(y-t^3)=3t^2(x-t)$ .....(1)

now solve the above equation with

$y=x^3$ .....(2)

By (1) & (2)

$x^3-t^3=3t^2(x-t)$

$x^2+xt+t^2=3t^2$

$x^2+xt-2t^2=0$

$(x-t)(x+2t)=0$

$\Rightarrow x=-2t$ ==> Q $(-2t,-8t^3)$

Ordinate of required point = $\frac{2t^3+(-8t^3)}{3}=2t^3$

Q.13 (2)

$f'(x)=(2x-1)(x-\sin x)$

==> $f'(x)\ge 0$ in $x\in\left[\frac{1}{2},\infty\right]$

and $f'(x)\le 0$ in $x\in\left[-\infty,\frac{1}{2}\right]$

Q.14 **(2)**

$f(x)=(4a-3)(x+\log_e 5)+(a-7)\sin x$

$f(x)=(4a-3)(1)+(a-7)\cos x=0$

$\Rightarrow \cos x=\frac{3-4a}{a-7}$

$-1\le\frac{3-4a}{a-7}<1$

$\frac{3-4a}{a-7}+1\ge 0$ $\qquad$ $\frac{3-4a}{a-7}<1$

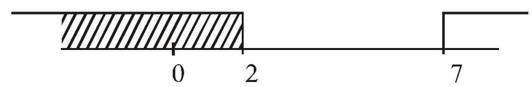
$\frac{3-4a+a-7}{a-7}\ge 0$ $\qquad$ $\frac{3-4a}{a-7}-1<0$

$\frac{-3a-4}{a-7}\ge 0$ $\qquad$ $\frac{3-4a-a+7}{a-7}<0$

$\frac{3a+4}{a-7}\le 0$ $\qquad$ $\frac{-5a+10}{a-7}<0$

$\frac{5a-10}{a-7}>0$

$\frac{5(a-2)}{a-7}>0$



$\alpha\in\left[-\frac{4}{3},2\right)$

Check end point $\left[-\frac{4}{3},2\right)$

Q.15 **[406]**

$y(x)=\int_0^x(2t^2-15t+10)\,dt$

$y'(x)\big]_{x=a}=\left[2x\ \ -15x+10\right]_a$ = 2a2 – 15a + 10

Slope of normal = $-\frac{1}{3}$

$\Rightarrow 2a^2-15a+10=3\Rightarrow a=7$

& $a=\frac{1}{2}$ (rejected)

$b=y(7)=\int_0^7(2t^2-15t+10)dt$

$=\left[\frac{2t^3}{3}\ \ \frac{15t^2}{2}\ \ 10t\right]_0^7$

$\Rightarrow 6b=4\times 7^3-45\times 49+60\times 7$

$|a+6b|=406$

Q.16 **[481]**

$f(x)=\sin\left(\cos^{-1}\left(\frac{1-2^{2x}}{1+2^{2x}}\right)\right)$ at x= 1 ; $2^{2x}=4$

for $\sin\left(\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right)$;

Let $\tan^{-1}x=\theta$; $\theta\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\therefore\ \sin(\cos^{-1}\cos 2\ )\sin 2-\theta=\theta$

$\left\{\begin{array}{l}\text{If}\quad x>1\Rightarrow\frac{\pi}{2}>\theta>\frac{\pi}{4}\\ \therefore\qquad \pi>2\theta>\ \frac{\pi}{2}\end{array}\right\}$

$= 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

$= \dfrac{2x}{1+x^2}$

Hence, $f(x) = \dfrac{2\cdot 2^x}{1+2^{2x}}$

$\therefore\ f'(x) = \dfrac{(1+2^{2x})(2.2^x\ln 2) - 2^{2x}\cdot 2\cdot \ln 2\cdot 2\cdot 2^x}{(1+2^{2x})}$

$\therefore\ f^1(x) = \dfrac{20\ln 2 - 32\ln 2}{25} = -\dfrac{12}{25}\ln 2$

So, a = 25, b = 12 $\Rightarrow |a^2 - b^2| = 25^2 - 12^2$

$= 625 - 144$

$= 481$

Q.17 (2)

$$f(x) = \begin{cases} -x\left(2-\sin\left(\frac{1}{x}\right)\right) & x<0 \\ 0 & x=0 \\ x\left(2-\sin\left(\frac{1}{x}\right)\right) & \end{cases}$$

$$f'(x) = \begin{cases} -\left(2-\sin\frac{1}{x}\right) - x\left(-\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)\right) & x<0 \\ \left(2-\sin\frac{1}{x}\right) + x\left(-\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)\right) & x>0 \end{cases}$$

$$f'(x) = \begin{cases} -2+\sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} & x<0 \\ 2-\sin\frac{1}{x}\ \ \frac{1}{x}\cos\frac{1}{x} & x>0 \end{cases}$$

f'(x) is an oscillating function which is non-monotonic in $(-\infty, 0) \cup (0, \infty)$.

Option (2)

Q.18 [5]

$f : [-1, 1] \to R$

$f(x) = ax^2 + bx + c$

$f(-1) = a - b + c = 2$ ...(1)

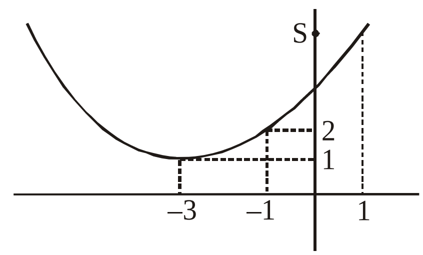
$f'(-1) = -2a + b = 1$ ...(2)

$f''(x) = 2a$

$\Rightarrow$ Max. value of $f''(x) = 2a = \dfrac{1}{2}$

$\Rightarrow a = \dfrac{1}{4};\quad b = \dfrac{3}{2};\quad c = \dfrac{13}{4}$

$\therefore\ f(x) = \dfrac{x^2}{4} + \dfrac{3}{2}x + \dfrac{13}{4}$



For, $x \in [-1, 1] \Rightarrow 2 \le f(x) \le 5$

$\therefore$ Least value of α is 5

Q.19 [8]

Let $p'(x) = a(x-1)(x+1) = a(x^2-1)$

$p(x) = a\int (x^2) - 1\ dx + c$

$= a\left(\dfrac{x^3}{3} - x\right) + c$

Now $p(-3) = 0$

$\Rightarrow a\left(-\dfrac{27}{3} + 3\right) + c = 0$

$\Rightarrow -6a + c = 0$ ...(1)

Now $\displaystyle\int_{-1}^{1}\left(a\left(\frac{x^3}{3} - x\right) + c\right)dx = 18$

$= 2c = 18 \Rightarrow c = 9$ ...(2)

$\Rightarrow$ from (1) & (2) $\Rightarrow -6a + 9 = 0 \Rightarrow a = \dfrac{3}{2}$

$\Rightarrow p(x) = \dfrac{3}{2}\left(\dfrac{x^3}{3} - x\right) + 9$

sum of coefficient

$= \dfrac{1}{2} - \dfrac{3}{2} + 9$

$= 8$

Q.20 (3)
Q.21 (4)
Q.22 [3]
Q.23 [1]
Q.24 (1)
Q.25 (3)
Q.26 (1)
Q.27 (3)
Q.28 (2)
Q.29 [22]
Q.30 (3)
Q.31 [2]

Q.32 (3)
Q.33 [36]
Q.34 (2)
Q.35 [72]
Q.36 **(3)**

JEE-ADVANCED PREVIOUS YEAR'S

Q.1 A → s; B → t; C → r; D → r.

(A) $\text{Re}\left(\dfrac{2i(x+iy)}{1-(x^2-y^2+2xyi)}\right)$

$= \text{Re}\left(\dfrac{-2y+2ix}{1-x^2+y^2-2xyi)}\right)$

$= \text{Re}\left(\dfrac{-2y+2ix}{2y(y-ix)}\right) = \text{Re}\,(-1/y) = \dfrac{-1}{y}$

$= -1 \le y \le 1 \quad = \dfrac{-1}{y} \ge 1 \text{ or } \dfrac{-1}{y} \le -1$

2nd Method

$\text{Re}\left(\dfrac{2ie^{i\theta}}{1-e^{2i\theta}}\right) = \text{Re}\left(\dfrac{2i(\cos\theta+i\sin\theta)}{1-(\cos 2\theta+i\sin 2\theta)}\right)$

$= \text{Re}\left(\dfrac{2i(\cos\theta+i\sin\theta)}{2\sin^2\theta-2i\sin\theta\cos\theta)}\right)$

$= \text{Re}\left(\dfrac{i(\cos\theta+i\sin\theta)}{\sin\theta(\sin\theta-i\cos\theta)}\right)$

$= \text{Re}\left(\dfrac{(\cos\theta+i\sin\theta)}{-\sin\theta(\cos\theta+i\sin\theta)}\right)$

$= \text{Re}\left(\dfrac{-1}{\sin\theta}\right)$

as $-1 \le \sin\theta \le 1$
$(-\infty, 0) \cup (0, \infty)$

(B) $-1 \le \dfrac{8.3^{x-2}}{1-3^{2x-2}} \le 1$

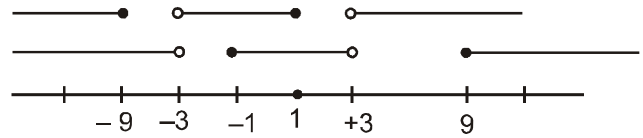
$\Rightarrow \quad -1 \le \dfrac{8t}{9-t^2} \le 1$

$\Rightarrow \quad -1 \le \dfrac{8t}{9-t^2} \le 1$

$\Rightarrow \quad 0 \le \dfrac{9-t^2+8t}{9-t^2} \cap \dfrac{8t}{9-t^2} - 1 \le 0$

$\Rightarrow \quad 0 \le \dfrac{t^2-8t-9}{t^2-9} \cap \dfrac{8t-9+t^2}{9-t^2} \le 0$

$\Rightarrow \quad 0 \le \dfrac{(t-9)(t+1)}{(t-3)(t+3)} \cap \dfrac{(t+9)(t-1)}{(t-3)(t+3)} \ge 0$



$\Rightarrow \quad t \in (-\infty, -9] \cup [-1, 1] \cup [9, \infty)$
$\Rightarrow \quad x \in (-\infty, 0) \cup [2, \infty)$

(C) $f(\theta) = 2\sec^2\theta$
$\Rightarrow \quad f(\theta) \ge 2$
$\Rightarrow \quad f(\theta) \in [2, \infty)$

(D) $f(x) = x^{3/2}(3x-10)$

$\Rightarrow \quad f'(x) = x^{3/2}\, 3 + \dfrac{3}{2}x^{1/2}(3x-10)$

as $f'(x) \ge 0$

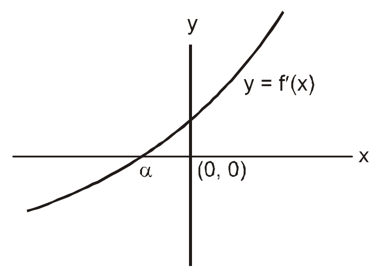
$\Rightarrow \quad x^{1/2}\left[3x+\dfrac{3}{2}(3x-10)\right] \ge 0$

$\Rightarrow \quad 3x + \dfrac{9x}{2} - 15 \ge 0$

$\Rightarrow \quad \dfrac{15x}{2} - 15 \ge 0$

$\Rightarrow \quad x \ge 2$
$\Rightarrow \quad x \in [2, \infty)$

Q.2 [2]
$f(x) = x^4 - 4x^3 + 12x^2 + x - 1$
$f'(x) = 4x^3 - 12x^2 + 24x + 1$



$f''(x) = 12x^2 - 24x + 24$
$= 12(x^2 - 2x + 2) > 0 \quad \forall\ x \in R$
$\therefore$ $f'(x)$ is S.I. function
Let α is a real root of the eqution $f'(x) = 0$
$\therefore$ f(x) is MD for $x \in (-\infty, \alpha)$ and
M.I. for $x \in (\alpha, \infty)$ where $\alpha < 0$
$\therefore$ $f(0) = -1$ and $\alpha < 0$
$\Rightarrow$ $f(\alpha)$ is also negative
$\therefore$ $f(x) = 0$ has two real & distinct roots.

Q.3 [2]

$\because \quad \Delta_2 = \dfrac{\Delta_1}{2}$ (by property)

$\because \quad \dfrac{\Delta_1}{\Delta_2} = 2$

Q.4 (A,BD)

Equation of normal is

$y = mx - 2m - m^3$

(9, 6) satisfies it

$6 = 9m - 2m - m^3$

$m^3 - 7m + 6 = 0 \quad \Rightarrow \quad m = 1, 2, -3$

$m = 1 \quad \Rightarrow \quad y = x - 3$

$m = 2 \quad \Rightarrow \quad y = 2x - 12$

$m = -3 \quad \Rightarrow \quad y = -3x + 33$

Q.5 (B)

Equation of normal at P(6, 3)

$$\frac{a^2x}{6} + \frac{b^2y}{3} = a^2 + b^2$$

It passes through (9, 0)

$$\frac{3}{2}a^2 = a^2 + b^2 \quad \Rightarrow \frac{3}{2} = \frac{a^2+b^2}{a^2} = 1 + \frac{b^2}{a^2}$$

$$\Rightarrow e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{\frac{3}{2}}$$

Q.6 (AB)

Slope of tangents = 2

Equation of tangents $y = 2x \pm \sqrt{9.4-4}$

$\Rightarrow y = 2x \pm \sqrt{32}$

$\Rightarrow 2x - y \pm 4\sqrt{2} = 0$ ....(i)

Let point of contact be (x_1, y_1)

then equation (i) will be identical to the equation

$$\frac{xx_1}{9} - \frac{yy_1}{4} - 1 = 0$$

$$\therefore \frac{x_1/9}{2} = \frac{y_1/4}{1} = \frac{-1}{\pm 4\sqrt{2}}$$

$$\Rightarrow (x_1, y_1) \equiv \left(-\frac{9}{2\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \text{ and } \left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Q.7 [9]

$p'= \lambda(x-1)(x-3) = \lambda(x^2 - 4x + 3)$

$p(x) = \lambda(x^3/3 - 2x^2 + 3x) + \mu$

$p(1) = 6$

$6 = \lambda(1/3 - 2 + 3) + \mu$

$6 = \lambda(1/3 + 1) + \mu$

$18 = 4\lambda + 3\mu$ ...(i)

$p(3) = 2$

$2 = \lambda(27/3 - 2 \times 9 + 9) + \mu$

$2 = \mu$

$\mu = 2 \Rightarrow \lambda = 3$

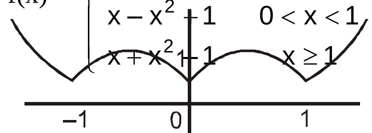
$p'(x) = 3(x-1)(x-3)$

$p'(0) = 3(-1)(-3) = 9$

Q.8 [5]

$f(x) = |x| + |x^2 - 1|$

$$f(x) = \begin{cases} -x + x^2 - 1 & x < -1 \\ -x - x^2 + 1 & -1 \le x \le 0 \\ x - x^2 + 1 & 0 < x < 1 \\ x + x^2 - 1 & x \ge 1 \end{cases}$$



$$f(x) = \begin{cases} x^2 - x - 1 & x < -1 \\ -x^2 - x + 1 & -1 \le x \le 0 \\ -x^2 + x + 1 & 0 < x < 1 \\ x^2 + x - 1 & x \ge 1 \end{cases}$$

Q.9 (C)

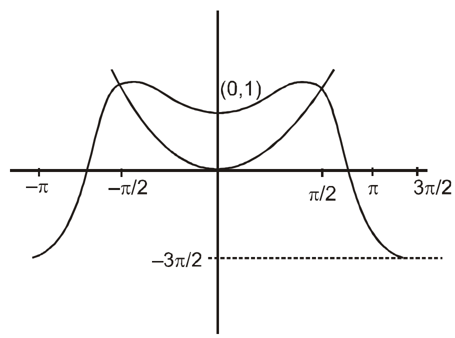
$x^2 = x \sin x + \cos x$

$f(x) = x^2$

$g(x) = x \sin x + \cos x$

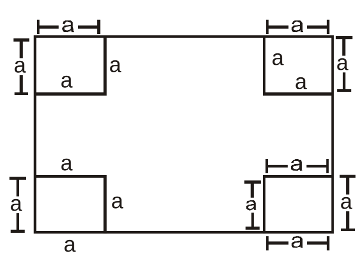
$g'(x) = \sin x + x\cos x - \sin x$

$g'(x) = x \cos x$



Only two solution.

Q.10 (A,C)



Let $\ell = 8x$, $b = 15x$

$\therefore$ Volume $= (8x - 2a)(15x - 2a)(a) = 4a^3 - 46a^2x + 120ax^2$

$$\frac{dV}{da} = 6a^2 - 46ax + 60x^2$$

$$\left(\frac{dV}{da}\right)_{\text{at } x=5} = 0$$

$\therefore \quad x = 3$ and $\frac{5}{6}$

$$\frac{d^2V}{da^2} = 6a - 23x$$

$$\left(\frac{d^2V}{da^2}\right)_{at\ a=5\ \&\ x=3} < 0,$$

So, at x = 3 gives maxima

$$\left(\frac{d^2V}{da^2}\right)_{at\ a=5\ \&\ x=\frac{5}{6}} > 0$$

So, at $x = \frac{5}{6}$ gives minima.

$\frac{dV}{da} = 0$ when a = 5 given ($\therefore\ 4a^2 = 100$ given for maximum volume)

at a = 5

by $\frac{dV}{da} = 0$

$\Rightarrow \quad 6x^2 - 23x + 15 = 0$

$x = 3$ or $5/6$

So by x = 3 (for max volume)

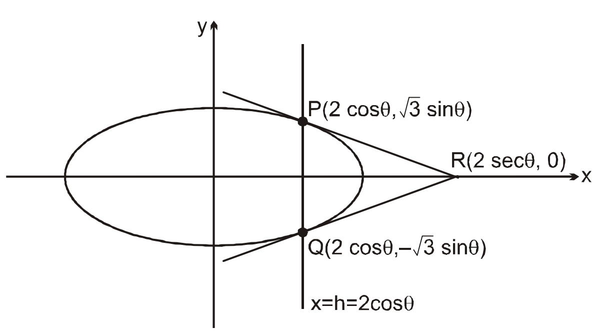
$8x = 24, \quad 15x = 45$ **Ans. (A, C)**

Q.11 [9]

Point of intersection of tangents at P and Q is R(2 secθ, 0)

Area of $\Delta PQR = \frac{1}{2} \cdot 2\sqrt{3}\ \sin\theta \cdot (2\sec\theta - 2\cos\theta)$

$\Rightarrow \quad \Delta = 2\sqrt{3} \cdot \frac{\sin^3\theta}{\cos\theta}$; where $\cos\theta \in \left[\frac{1}{4}, \frac{1}{2}\right]$



Now $\frac{d\Delta}{d\theta} = \frac{2\sqrt{3}\left[\cos\theta.3\sin^2\theta\cos\theta - \sin^3\theta(-\sin\theta)\right]}{\cos^2\theta} > 0$

As θ increases, Δ increases ⇒ when cosθ decreases, Δ increases

$\therefore \quad \Delta_{min.}$ occurs at cosθ = 1/2, Therefore $\Delta_2 =$

$$2\sqrt{3} \cdot \frac{(1-1/4)^{3/2}}{1/2} = 4\sqrt{3} \cdot \frac{3\sqrt{3}}{8} = \frac{36}{8}$$

$\Delta_{max.}$ occurs at cosθ = 1/4, Therefore $\Delta_1 =$

$$2\sqrt{3} \cdot \frac{(1-1/16)^{3/2}}{1/4} = 8\sqrt{3} \cdot \frac{15.\sqrt{15}}{4.4.4} = \frac{2\sqrt{3}.15.\sqrt{3}\sqrt{5}}{16}$$

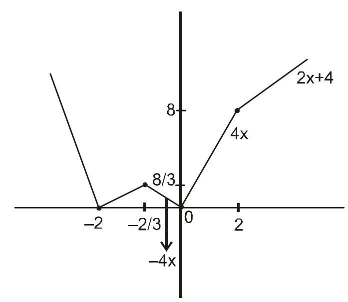
$$\Rightarrow \quad \Delta_1 = \frac{45}{8}\sqrt{5}$$

Now $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45 - 36 = 9$

Q.12 (A, B)

$f(x) = 2|x| + |x + 2| - |\,|x + 2| - 2|x|\,|$

$$= \begin{cases} -2x-4 & x \le -2 \\ 2x+4 & -2 < x \le -2/3 \\ -4x & -2/3 < x \le 0 \\ 4x & 0 < x \le 2 \\ 2x+4 & x > 2 \end{cases}$$



Graph of y = f(x) is

minima at x = –2, 0 ; maxima at x = –2/3

Comprehension (Q. No. 13 & 14)

Q.13 (D)

$f''(x) - 2f'(x) + f(x) \ge e^x$

$f''(x)\cdot e^{-x} - f'(x)e^{-x} - f'(x)e^{-x} + f(x)e^{-x} \ge 1$

$\frac{d}{dx}(f'(x)e^{-x}) - \frac{d}{dx}(f(x)\cdot e^{-x}) \ge 1$

$\frac{d}{dx}(f'(x)\,e^{-x} - f(x)\,e^{-x}) \ge 1$

$\Rightarrow \frac{d^2}{dx^2}(e^{-x}f(x)) \ge 1\ \forall\ x \in [0, 1]$

Let $\phi(x) = e^{-x} f(x)$

$\Rightarrow \phi(x)$ is concave upward

$f(0) = f(1) = 0$

$\Rightarrow \phi(0) = 0 = \phi(1) \qquad \Rightarrow \phi(x) < 0$

$\Rightarrow f(x) < 0$

Q.14 (C)

$\phi'(x) < 0,\ x \in (0, 1/4)$

and

$\phi'(x) > 0,\ x \in (1/4, 1)$

$\Rightarrow e^{-x} f'(x) - e^{-x} f(x) < 0,\ x \in (0, 1/4)$

$f'(x) < f(x),\ 0 < x < 1/4$

Q.15 **(A)**

tangent at F $\quad yt = x + 4t^2$

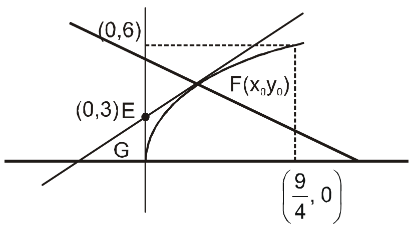
a : x = 0 y = 4t $\quad (0, 4t)$

$(4t^2, 8t)$ satisfies the line

$8t = 4mt^2 + 3$

$4mt^2 - 8t + 3 = 0$

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 0 & 3 & 1 \\ 0 & 4t & 1 \\ 4t^2 & 8t & 1 \end{vmatrix} = \frac{1}{2}(4t^2(3-4t))$$

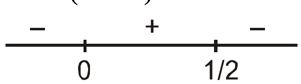


$= 2t^2(3 - 4t)$

$A = 2[3t^2 - 4t^3]$

$$\frac{dA}{dt} = 2[6t - 12t^2]$$

$= 24\,t(1 - 2t)$



t = 1/2 maxima

G(0, 4t) $\Rightarrow$ G(0, 2)

$y_1 = 2$

$(x_0, y_0) = (4t^2, 8t) = (1, 4)$

$y_0 = 4$

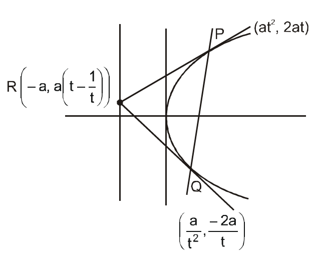
$$\text{Area} = 2\left(\frac{3}{4} - \frac{1}{2}\right) = 2\left(\frac{3-2}{4}\right) = \frac{1}{2}$$

Comprehension (Q. No. 16 & 17)

Q.16 **(B)**

R lies on y = 2x + a

R, y = 2x + a



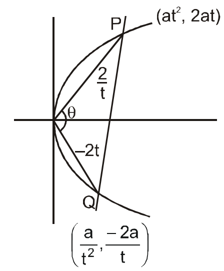
$$\Rightarrow \quad a\left(t - \frac{1}{t}\right) = -a$$

$$t - \frac{1}{t} = -1$$

$$\Rightarrow \quad \left(t + \frac{1}{t}\right)^2 = 1 + 4 = 5$$

$$PQ = a\left(t + \frac{1}{t}\right)^2 = 5a$$

Q.17 **(D)**



$$t - \frac{1}{t} = -1$$

$$\Rightarrow \quad t + \frac{1}{t} = \sqrt{5}$$

$$\tan\theta = \frac{\frac{2}{t} + 2t}{1 - 4} = \frac{2\left(\frac{1}{t} + t\right)}{-3} = \frac{2\sqrt{5}}{-3}$$

Q.18 **(BD)**

$f(x) = x^5 - 5x + a = 0$

$x^5 - 5x = -a$

$x(x^4 - 5) = -a$

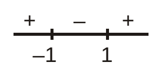
$x(x^2 - \sqrt{5})(x^2 + \sqrt{5}) = -a$

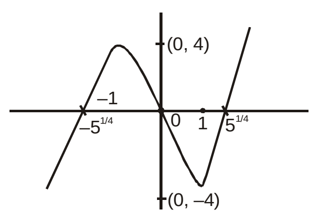
$x(x - 5^{1/4})(x + 5^{1/4})(x^2 + \sqrt{5}) = -a \qquad ...(1)$

$f'(x) = 5x^4 - 5 = 0$

$(x^2 - 1)(x^2 + 1) = 0$

$(x - 1)(x + 1)(x^2 + 1) = 0$





Q.19 [8]

$(y - x^5)^2 = x(1 + x^2)^2$

$$2(y - x^5)\left(\frac{dy}{dx} - 5x^4\right) = (1 + x^2)^2 + 2x(1 + x^2)2x$$

at point $(1, 3)$

$$\therefore\ 2(3 - 1)\left(\frac{dy}{dx} - 5\right) = 4 + 8$$

$$\frac{dy}{dx} - 5 = \frac{12}{4} = 3$$

$$\frac{dy}{dx} = 8$$

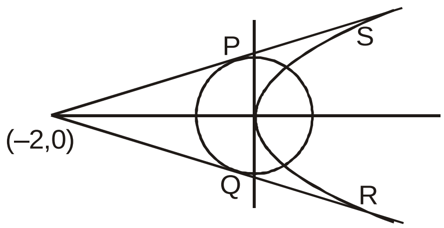
Q.20 (D)

$y = mx + \frac{2}{m}$

If it is tangent to $x^2 + y^2 = 2$

Then, $\left|\frac{\frac{2}{m}}{\sqrt{1+m^2}}\right| = \sqrt{2}$

$\Rightarrow \frac{4}{m^2(1+m^2)} = 2 \Rightarrow m = \pm 1.$



Hence equation of tangent is $y = x + 2$
and $y = -x - 2$.
Chord of contact PQ is $-2x = 2 \Rightarrow x = -1$
Chord of contanct RS is $y. 0 = 4(x - 2) \Rightarrow x = 2$
Hence co-ordinates of P, Q, R, S are (–1, 1) ; (–1, –1) ; (2, – 4) & (2, 4)
Area of trapezium is

$= \frac{1}{2}$ (PQ + RS) × Height $= \frac{1}{2}$ (10) × 3 = 15

Comprehension (Q. No.21 & 22)

Q.21 (D)

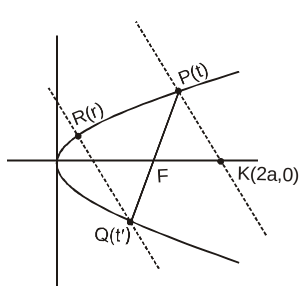
$m_{PK} = m_{QR}$

$$\frac{2at - 0}{at^2 - 2a} = \frac{2at' - 2ar}{a(t')^2 - ar^2}$$

$$\frac{t}{t^2 - 2} = \frac{t' - r}{(t')^2 - r^2}$$

$- t' - tr^2 = - t - rt^2 - 2t' + 2r$, $tt' = -1$
$t' - tr^2 = - t + 2r - rt^2$
$-tr^2 + r(t^2 - 2) + t' + t = 0$

$$\lambda = \frac{(2 - t^2) \pm \sqrt{(t^2 - 2)^2 + 4(-1 + t^2)}}{-2t}$$



$$= \frac{(2 - t^2) \pm \sqrt{t^4}}{-2t} = \frac{2 - t^2 \pm t^2}{-2t}$$

$r = -\frac{1}{t}$

It is not possible as the R & Q will be one same.

or $r = \frac{t^2 - 1}{t}$ **(D) Ans.**

Q.22 (B)

Tangent at P is $ty = x + at^2$
Normal at S is $y + sx = 2as + as^2$

$ty + x = 2a + \frac{a}{t^2}$

$ty = 2a + \frac{a}{t^2} - ty + at^2$

$2t^3y = at^4 + 2at^2 + a$

$y = \frac{a(t^2 + 1)^2}{2t^3}$.

Q.23 [4]

Let any point on $y^2 = 4x$ be $\left(\frac{p^2}{4}, p\right)$, where 'p' is a parameter.

Let the image of the above point on curve 'C' be (h, k).

The slope of the line $x + y + 4 = 0$ is –1.

Hence, slope of the line joining $\left(\frac{p^2}{4}, p\right)$ and (h, k) is 1.

$\Rightarrow \quad k - p = h - \frac{p^2}{4}$

$\Rightarrow \quad \frac{p^2}{4} - p - h + k = 0$ .... (1)

Also, the midpoint of $\left(\frac{p^2}{4}, p\right)$ and (h, k) lies on x + y + 4 = 0

$\Rightarrow \quad \frac{\frac{p^2}{4} + h}{2} + \frac{p + k}{2} + 4 = 0$

$\Rightarrow \quad \frac{p^2}{4} + p + h + k + 8 = 0$ .... (2)

Adding (1) and (2) we get,

$\frac{p^2}{2} + 2k + 8 = 0$

$\Rightarrow \quad p^2 = -4k - 16$ .... (3)

Subtracting (1) from (2) we get,

$2h + 2p + 8 = 0$

$\Rightarrow \quad p = -4 - h$ .... (4)

Eliminating 'p' from (3) and (4), we get

$(h + 4)^2 = -4(k + 4)$.

Hence, equation of curve C is $(x + 4)^2 = -4(y + 4)$

For $y = -5$, we get $(x + 4)^2 = 4 \Rightarrow x = -6, -2$

Hence, the distance between A and B is 4.

Q.24 [2]

Slope of tangents at the end points of the latusrectum of $y^2 = 4x$ are ± 1

Hence, the slopes of normals are ∓ 1.

Equations of normals are $y - 2 = -1(x - 1)$ and $y + 2 = 1(x - 1)$

$x + y - 3 = 0$ and $x - y - 3 = 0$

If these are tangents to the circle

$(x - 3)^2 + (y + 2)^2 = r^2$,

the distance of the centre of the circle (3, –2) from the tangents is equal to 'r'.

$\Rightarrow \left|\dfrac{3+(-2)-3}{\sqrt{1^2+1^2}}\right| = r \quad \Rightarrow r = \sqrt{2} \quad \Rightarrow r^2 = 2$

Q.25 (**A**,D)

Let the coordinates of P and Q be

$\left(\dfrac{t_1^2}{2}, t_1\right)$ and $\left(\dfrac{t_2^2}{2}, t_2\right)$ respectively.

Since PQ is the diameter of the circle on which the vertex O lies,

$OP \perp OQ \Rightarrow \dfrac{2}{t_1} \times \dfrac{2}{t_2} = -1 \Rightarrow t_1t_2 = -4$

Now area of $\Delta OPQ = \dfrac{1}{2} OP.OQ = 3\sqrt{2}$

$\Rightarrow \dfrac{1}{2}\sqrt{\dfrac{t_1^4}{4}+t_1^2}\sqrt{\dfrac{t_2^4}{4}+t_2^2} = 3\sqrt{2}$

$\Rightarrow \dfrac{1}{2}t_1t_2\sqrt{\dfrac{4+t_1^2}{4}}\sqrt{\dfrac{4+t_2^2}{4}} = 3\sqrt{2}$

Squaring both sides, we get

$4 \times \dfrac{16+4(t_1^2+t_2^2)+16}{16} = 18$

$\Rightarrow t_1^2 + t_2^2 = 10$

$\Rightarrow t_1^4 - 10t_1^2 + 16 = 0$

$\Rightarrow t_1^2 = 2, 8$

Hence, coordinates of $P\left(\dfrac{t_1^2}{2}, t_1\right)$ are

$\left(1, \sqrt{2}\right)$ or $\left(4, 2\sqrt{2}\right)$

Q.26 [4]

$e^2 = 1 - \dfrac{b^2}{a^2} = 1 - \dfrac{5}{9} = \dfrac{4}{9}$

$e = \dfrac{2}{3}$ focii (2, 0) (–2, 0)

$p_1 : y^2 = 8x$,

$y = m_1x + \dfrac{2}{m_1}$

$0 = -4m_1 + \dfrac{2}{m_1}$

$\Rightarrow \quad 4m_1^2 = 2$

$\Rightarrow \quad m_1 = \pm\dfrac{1}{\sqrt{2}}$

$p_2 : y^2 = -16x$

$\Rightarrow \quad y = m_2x - \dfrac{4}{m_2}$

$\Rightarrow \quad 0 = 2m_2 - \dfrac{4}{m_2}$

$\Rightarrow \quad 2m_2^2 = 4$

$\dfrac{1}{m_1^2} + m_2^2 = 2 + 2 = 4$

Q.27 (A,B)

$E_1 \rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$E_2 = \dfrac{x^2}{A^2} \; \dfrac{y^2}{B^2} = 1$

Now as $x + y = 3$ is a tangent

$a^2 + b^2 = A^2 + B^2 = 9$

Now point P is

$x^2 + (2 - x)^2 = 2$

$2x^2 - 4x + 2 = 0$

$x = 1$

so P is (1, 2)

points Q & R are $\left(\dfrac{5}{3}, \dfrac{4}{3}\right)$ & $\left(\dfrac{1}{3}, \dfrac{8}{3}\right)$

Now $\left(\dfrac{5}{3}, \dfrac{4}{3}\right)$ lies on E_1 so $\dfrac{25}{9a^2} + \dfrac{16}{9(9-a^2)} = 1$

$\Rightarrow \quad 225 - 25a^2 + 16a^2 = 9a^2(9 - a^2)$

$\Rightarrow \quad 225 - 9a^2 = 9a^2(9 - a^2)$

$\Rightarrow \quad 25 - a^2 = a^2(9 - a^2)$

$\Rightarrow \quad a^4 - 10a^2 + 25 = 0 \Rightarrow a^2 = 5$ so $b^2 = 4$

$e_1^2 = \dfrac{1}{5}$

Now $\left(\dfrac{1}{3}, \dfrac{8}{3}\right)$ lies on E_2

$$\frac{1}{A^2} + \frac{64}{(9 - A^2)} = 9$$

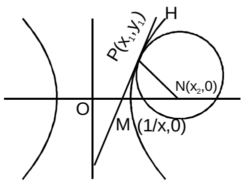
$9 - A^2 + 64A^2 = 9A^2\,(9 - A^2)$

$1 + 7A^2 = A^2 = 9A^2 - A^4 \quad \Rightarrow A^4 - 2A^2 + 1 = 0$

$\Rightarrow A^2 = 1 \qquad$ So $\qquad B^2 = 8$

$$e_2^{\,2} = \frac{7}{8}$$

Q.28 (A,B,D)



Equation tangent to H at P is $xx_1 - yy_1 = 1$

$$\ell = \frac{x_1 + x_2 + \frac{1}{x_1}}{3}, \; m = \frac{y_1}{3} = \frac{\sqrt{x_1^2 - 1}}{3}$$

now, $$\left.\frac{dy}{dx}\right|_{H \text{ at } P} = \left.\frac{dy}{dx}\right|_{S \text{ at } P}$$

$$\Rightarrow \frac{x_1}{y_1} = \frac{x_2 - x_1}{y_1} \quad \Rightarrow x_2 = 2x_1$$

So , $$\ell = x_1 + \frac{1}{3x_1}$$

$$\frac{d\ell}{dx_1} = 1 - \frac{1}{3x_1^2}, \quad \frac{dm}{dy_1} = \frac{1}{3}, \quad \frac{dm}{dx_1} = \frac{1}{3}.\frac{x_1}{\sqrt{x_1^2 - 3}}$$

Q.29 [4]

Volume of material $V = \pi r^2 h$

$\Rightarrow \quad V_1 = \pi(r + 2)^2 2 + \pi(r + 2)^2 h - \pi r^2 h$

$\Rightarrow \quad V_1 = 2\pi(r + 2)^2 + \pi h(4 + 4r)$

$\Rightarrow \quad V_1 = 2\pi(r + 2)^2 + 4\pi h(r + 1)$

$$\Rightarrow \quad V_1 = 2\pi\left((r+2)^2 + \frac{2(r+1)V}{\pi r^2}\right)$$

$$\Rightarrow \quad \frac{dV_1}{dr} = 2\pi\left(2(r+2) + \frac{2v}{\pi}\left(\frac{-1}{r^2} - \frac{2}{r^3}\right)\right) = 0$$

$$\Rightarrow \quad 24 + \frac{2V}{\pi}\left(\frac{-2-10}{10^3}\right) = 0$$

$$\Rightarrow \quad \frac{24v}{10^3\pi} = 24 \quad \Rightarrow v = 10^3\pi$$

$$\Rightarrow \quad \frac{v}{250\pi} = 4$$

Q.30 [3]

$$F(x) = \int_{x}^{x^2 + \frac{\pi}{6}} 2\cos^2 + dt \quad ;$$

$$F'(x) = 2\left(\cos\left(x^2 + \frac{\pi}{6}\right)\right)^2 2x - 2\cos^2 x$$

$$\because \quad F'(a) + 2 = \int_0^a f(x)dx$$

$$\Rightarrow 2\left(\cos\left(a^2 + \frac{\pi}{6}\right)\right)^2 2a - 2\cos^2 a + 2 = \int_0^a f(x)dx$$

$$\Rightarrow 4\cos^2\left(a^2 + \frac{\pi}{6}\right) + 4a\ 2\cos$$

$$\left(a^2 + \frac{\pi}{6}\right).\left(-\sin\left(a^2 + \frac{\pi}{6}\right)\right) \times 2a + 4\cos a \sin a = f(a)$$

$$\therefore f(0) = 4\left(\frac{\sqrt{3}}{2}\right)^2 = 3$$

Q.31 **(B, C)**

Let $h(x) = f(x) - 3g(x)$

$\left.\begin{matrix} h(-1) = 3 \\ h(0) = 3 \end{matrix}\right\} \Rightarrow h'(x) = 0$ has atleast one root in $(-1, 0)$ and atleast one root in $(0, 2)$

$h(2) = 3$

But since $h''(x) = 0$ has no root in $(-1, 0)$ & $(0, 2)$ therefore $h'(x) = 0$ has exactly 1 root in $(-1, 0)$ & exactly 1 root in $(0, 2)$

Q.32 **(C,D)**

$e^x \in (1,e)$ for $x \in (0,1)$

and $$0 < \int_0^x f(t)\sin t\ dt < 1 \text{ in } (0,1)$$

$\Rightarrow$ (A) is wrong

$$f(x) + \int_0^{x/2} f(t)\sin t\ dt < 0$$

$\Rightarrow$ (B) is wrong

Let $$g(x) = x - \int_0^{\frac{\pi}{2} - x} f(t)\cos t\ dt$$

$$\Rightarrow \quad g(0) = -\int_0^{\frac{\pi}{2}} f(t)\cos t\, dt < 0$$

$$g(1) = 1 - \int_0^{\frac{\pi}{2}-1} f(t)\cos t\, dt < 0$$

$\Rightarrow$ (C) is correct

Let $h(x) = x^9 - f(x)$

$h(0) = -f(0) < 0$

$h(1) = 1 - f(1) > 0$

$\Rightarrow$ (D) is correct

Q.33 **(D)**

Q.34 **(C)**

Q.35 **(C)**

$f(x) = x + \ell nx - x\,\ell nx$

$$f'(x) = 1 + \frac{1}{x} - \ell nx - x\left(\frac{1}{x}\right) - \ell nx$$

$$f''(x) = -\frac{1}{x^2} - \frac{1}{x} < 0 \;\forall x \in (0,\infty)$$

$\therefore$ f'(x) is strictly decreasing function for $x \in (0, \infty)$

$$\left.\begin{array}{l}\lim\limits_{x\to\infty} f'(x) = -\infty \\ \lim\limits_{x\to 0^+} f'(x) = -\infty\end{array}\right\} \Rightarrow \qquad f'(x) = 0 \text{ has only}$$

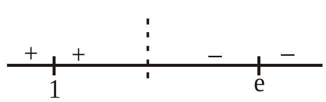
one real in $(0, \infty)$

$f'(1) = 1 > 0$

$$f'(e) = \frac{1}{e} - 1 < 0$$

$\therefore$ f'(x) = 0 has one root in (1, e)

Letf'(α) = 0 where $\alpha \in (1, e)$



$\therefore$ f(x) is increasing in (0, α) and decreasing in (α, ∞)

$f(1) = 1$ and $f(e^2) = e^2 + 2 - 2e^2 = 2 - e^2 < 0$

$\Rightarrow$ f(x) = 0 has one root in $(1, e^2)$

From column 1 : I and II are correct.

From column 2 : ii, iii, and iv are correct.

From column 3 : P, Q, S are correct

Q.36 **(B)**

$f''(x) > 0$ for all $x \in R$, $f(1/2) = 1/2$, $f(1) = 1$

$\Rightarrow$ f'(x) increases

Let $g(x) = f(x) - x$, $x \in [1/2, 1]$

Then g'(x) = 0 has atleast one real root in (1/2,1)

f'(x) = 1 has atleast one real root in (1/2,1)

Hence f'(x) increases $\Rightarrow$ f'(1) > 1

Q.37 **(A,C)**

$f'(x) - 2(f(x) > 0$

$$\Rightarrow \quad \frac{d}{dx}\left(f(x).e^{-2x}\right) > 0 \;\Rightarrow\; g(x) = f(x).e^{-2x} \text{ is an}$$

increasing function.

for x > 0, $\qquad$ g(x) > g(0)

$\Rightarrow$ $f(x).\,e^{-2x} > 1$ $\qquad \Rightarrow f(x) > e^{2x}$

Now $f'(x) > 2f(x) > 2.e^{2x}$

$\therefore$ f (x) is an increasing function

Q.38 **(BC)**

$$f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$$

$= \cos 2x - \cos 2x(-\cos^2 x + \sin^2 x) + \sin 2x(-2\sin x\cos x)$

$f(x) = \cos 4x + \cos 2x$

$\therefore$ $f(x) = 2\cos^2 2x + \cos 2x - 1$

Let cos2x = t

$\Rightarrow f(x) = 2t^2 + t - 1$ and $t \in [-1,1]$

f(x) attains its minima at $t = -\frac{1}{4} \in [-1,1]$

f(x), $t = -\frac{1}{4} \in [-1,1]$

$$\therefore \quad f(x)\Big|_{\min} = \frac{2}{16} - \frac{1}{4} - 1 = \frac{-9}{8}$$

$\therefore$ $f(x)\Big|_{\max} = 2 + 1 - 1 = 2$........... (when cos2x = 1)

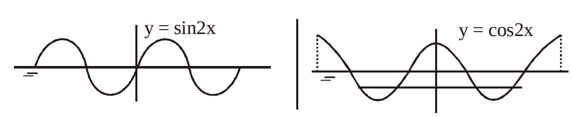
$f'(x) = -4\sin 4x - 2\sin 2x$

$f'(x) = 0 \Rightarrow 4\sin 4x + 2\sin 2x = 0$

$\Rightarrow 8\sin 2x\cos 2x + 2\sin 2x = 0$

$\Rightarrow 2\sin 2x(4\cos 2x + 1) = 0 \Rightarrow \sin 2x = 0$ or

$$\cos 2x = -\frac{1}{4}$$



Hence option (B), C)

Q.39 **(A,B,D)**

f(x) can't be constant throught the domain. Hence we can find $x \in (r, s)$ such that f(x) is one-one option (A) is true.

Option (B) : $|f'(x_0)| = \left|\dfrac{f(0)-f(-4)}{4}\right| \le 1$ (LMVT)

Option (C) : $f(x) = \sin(\sqrt{85}x)$ satisfies given condition

but $\lim\limits_{x\to\infty} \sin(\sqrt{85}x)$ D.N.E.

$\Rightarrow$ Incorrect

Option (D) : $g(x) = f^2(x) + (f'(x))^2$

$|f'(x_1)| \le 1$ (by LMVT)

$|f(x_1)| \le 2$ (given)

$\Rightarrow g(x_1) \le 5 \ \exists x_1 \in (-4, 0)$

Similarly

$g(x_2) \le 5 \quad \exists x_2 \in (0, 4)$

$g(0) = 85 \quad \Rightarrow g(x)$ has maxima in (x, x_2) say at α.

$g'(\alpha)\,(f(\alpha) + f''(\alpha))0 = 0$

If $f'(\alpha) = 0 \ \Rightarrow \ g(\alpha) = f^2(\alpha) \ge 85$ Not possible

$\Rightarrow f(\alpha) + f''(\alpha) = 0 \quad \exists\, \alpha \in (x_1, x_2) \in (-4, 4)$

option (D) correct.

Q.40 **(B, C)**

$f'(x) = (e^{(f(x)-g(x))})g'(x) \ \forall \ x \in \mathbb{R}$

$\Rightarrow e^{-f(x)}.f'(x) - e^{-g(x)} g'(x) = 0$

$\Rightarrow \int (e^{-f(x)}f'(x) - e^{-g(x)}.g'(x))dx = C$

$\Rightarrow -e^{-f(1)} + e^{-g(1)} = e^{-f(2)} + e^{-g(2)}$

$\Rightarrow -\dfrac{1}{e} + e^{-g(1)} = -e^{-f(2)} + \dfrac{1}{e}$

$\Rightarrow e^{-f(2)} + e^{-g(1)} = \dfrac{2}{e}$

$\therefore \ e^{-f(2)} < \dfrac{2}{e}$ and $e^{-g(1)} < \dfrac{2}{e}$

$\Rightarrow -f(2) < \ln 2 - 1$ and $-g(1) < \ln 2 - 1$

$\Rightarrow f(2) > 1 - \ln 2$ and $g(1) > 1 - \ln 2$

Q.41 (A,B,D)

$$f(x) = \begin{cases} (x+1)^5 - 2x, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \dfrac{2}{3}x^3 - 4x^2 + 7x - \dfrac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \dfrac{10}{3}, & x \ge 3 \end{cases}$$

for $x < 0$, $f(x)$ is continuous

& $\lim\limits_{x\to-\infty} f(x) = -\infty$ and $\lim\limits_{x\to 0^-} f(x) = 1$

Hence, $(-\infty, 1) \subset$ Range of f(x) in $(-\infty, 0)$

$f'(x) = 5(x+1)^4 - 2$, which changes sign in $(-\infty, 0)$

$\Rightarrow f(x)$ is non- monotonic in $(-\infty, 0)$

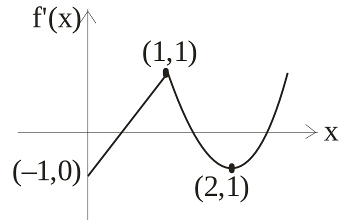
For $x \ge 3$, $f(x)$ is again continuous and

$\lim\limits_{x\to\infty} f(x) = \infty$ and $f(3) = \dfrac{1}{3}$

$\Rightarrow \left[\dfrac{1}{3}, \infty\right) \subset$ Range of $f(x)$ in $[3, \infty)$

Hence, range of $f(x)$ is R

$$f'(x) = \begin{cases} 2x - 1, & 0 \le x < 1 \\ 2x^2 - 8x + 7, & 1 \le x < 3 \end{cases}$$



Hence f' has a local maximum at $x = 1$ and f' is NOT differentiable at $x = 1$.

Q.42 (A,B,C)

$f(x) = (x-1)(x-2)(x-5)$

$F(x) = \int_0^x f(t)dt \ \ x > 0$

$F'(x) = f(x) = (x-1)(x-2)(x-5),\ x > 0$

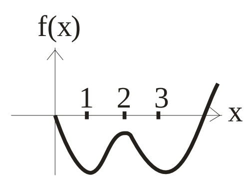
clearly F(x) has local minimum at x = 1,5

F(x) has local maximum at x = 2

$f(x) = x^3 = -8x^2 + 17x - 10$

$\Rightarrow F(x) = \int_0^x (t^3 - 8t^2 + 17t - 10)dt$

$F(x) = \dfrac{x^4}{4} - \dfrac{8x^3}{3} + \dfrac{17x^2}{2} - 10x$

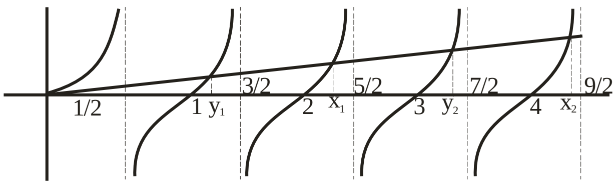


from the graph of y = F(x), clearly F(x) $0 \forall x \in (0,5)$

Q.43 (A,C,D)

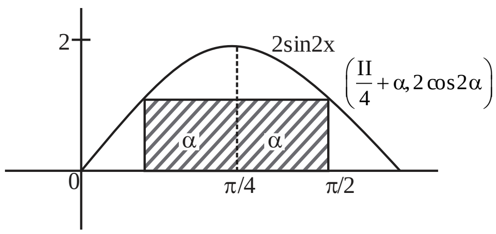
$f(x) = \dfrac{\sin \pi x}{x^2}$

$f'x = \dfrac{2x\cos\pi x\left(\dfrac{\pi x}{2} - \tan \pi x\right)}{x^4}$



$\Rightarrow |x_n - y_n| > 1$ for every n
$x_1 > y_1$
$x_n \in (2n, 2n + 1/2)$
$x_{n+1} - x_n > 2.$

Q.44 (C)



Perimeter = $2(2\alpha+2\cos 2\alpha)$
$P=4(\alpha+\cos 2\alpha)$

$$\frac{dP}{d\alpha}=4(1-2\sin 2\alpha)=0$$

$$\sin 2\alpha=\frac{1}{2}$$

$$2\alpha=\frac{\pi}{6},\frac{5\pi}{6}$$

$$\frac{d^2P}{d\alpha^2}=-4\cos 2\alpha$$

for maximum $\alpha=\frac{\pi}{12}$

Area = (2α) $(2\cos 2\alpha)$

$$=\frac{\pi}{6}\times 2\times\frac{\sqrt{3}}{2}=\frac{\pi}{2\sqrt{3}}$$

Q.45 [5.00]
$f(x)=(x^2-1)^2\ h(x)$; $h(x)=a_0+a_1x+a_2x^2+a_3x^3$
Now, $f(1) = f(-1)=0$
$\Rightarrow f'(\alpha)=0,\ a\in(-1, 1)$ [Rolle's Theorem]
Also, $f'(1)=f'(-1)=0 \Rightarrow f'(x)=0$ has atleast 3 root, –1, α, 1 with $-1<\alpha<1$
$\Rightarrow f''(x)=0$ will have at least 2 root, say β, γ such that $-1<\beta<\alpha<\gamma<1$ [Rolle's Theorem]
So, min $(m_{f''})=2$
and we find $(m_{f'}+m_{f''})=5$ for $f(x) = (x^2-1)^2$.
Thus Ans. 5

Q.46 (A,C)

$$f'(x)=\frac{f(x)}{b^2+x^2}$$

$$\int\frac{f'(x)}{f(x)}dx=\int\frac{dx}{x^2+b^2}$$

$$\Rightarrow \ell n\,|f(x)|=\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+c$$

Now $f(0)=1$

$$\therefore\ c=0\ \therefore\ |f(x)|=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

$$\Rightarrow\ f(x)=\pm e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

since $f(0)=1$ $\therefore$ $f(x)=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$
$x \to -x$

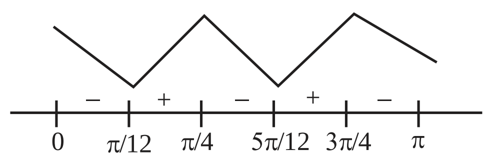
$$f(-x)=e^{-\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

$\therefore$ $f(-x)\,.\,f(x)=e^0=1$ (option C)
and for $b > 0$

$$f(x)=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

$\Rightarrow f(x)$ is increasing for all $x\in R$ (option A)

Q.47 [0.50]
$f(\theta)=(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^4$
$f(\theta)=\sin^2 2\theta-\sin 2\theta+2$
$f'(\theta)=2(\sin 2\theta).(2\cos 2\theta)-2\cos 2\theta$
$= 2\cos 2\theta(2\sin 2\theta-1)$
critical points



so, minimum at $\theta=\frac{\pi}{12},\frac{5\pi}{12}$

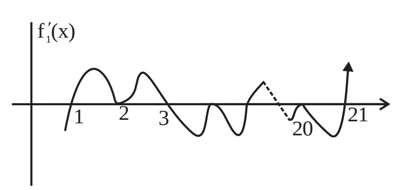
$$\lambda_1+\lambda_2=\frac{1}{12}+\frac{5}{12}=\frac{6}{12}=\frac{1}{2}$$

Question Stem for Question Nos. 48 and 49
Question Stem

Q.48 [57.00]

$$f_1(x)=\int_0^x\prod_{j=1}^{21}(t-j)^j\,dt$$

$$f_1'(x)=\prod_{j=1}^{21}(x-j)^j=(x-1)(x-2)^2(x-3)^3....(x-21)^{21}$$



So points of minima one 4m+1 wher m=0, 1, ... 5

$\Rightarrow$ m_1=6

Points of maxima are 4m–1 where m=1,2,...5 $\Rightarrow$ 5 $\Rightarrow$ n_1=5

$\Rightarrow$ $2m_1+3n_1+m_1n_1=57$

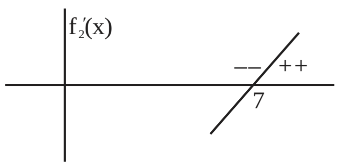
Q.49 [6.00]

$f_2'(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$

$\Rightarrow f_2'(x) = 2 \times 49 \times 50(x-1)^{49} - 50 \times 12 \times 49(x-1)^{48}$

$= 50 \times 49 \times 2(x-1)^{48}(x-1-6)$

$= 50 \times 49 \times 2(x-1)^{48}(x-7)$



Point of minima = 7

$\Rightarrow m_2 = 1$

No point of maxima

$\Rightarrow n_2 = 0$

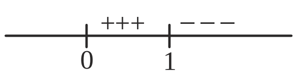
$6m_2 + 4n_2 + 8m_2n_2 = 6$

Paragraph

Q.50 (C)

$f'(x) = (|x| - x^2)e^{-x^2} + (|x| - x^2)e^{-x^2}$, $x \ge 0$

$f' = 2(x - x^2)e^{-x^2}$



hence option (D) is wrong

$g'(x) = xe^{-x^2}2x$

$f'(x) + g'(x) = 2 \times e^{-x^2}$

$f'(x) + g(x) = -e^{-x^2} + c$

$f'(x) + g(x) = -e^{-x^2} + 1$

$F(\ell n3) + g\left(\sqrt{\ell n3}\right) = 1 - \frac{1}{3} = \frac{2}{3}$ (option (A) is wrong)

$H(x) = \psi_1(x) - 1 - \alpha x = e^{-x} + x - 1 - \alpha x$,

$x \ge 1$ & $\alpha \in (1, x)$

$H(1) = e^{-1} + 1 - 1 - \alpha < 0$

$H'(x) = -e^{-x} + 1 - \alpha > 0 \Rightarrow H(x)$ is $\downarrow \Rightarrow$ option (B) is wrong

(C) $\psi_2(x) = 2(\psi_1(\beta) - 1)$

Applying L.M.V.T to $\psi_2(x)$ in [0, x]

$$\psi_2'(\beta) = \frac{\psi_2(x) - \psi_2(0)}{x}$$

$$2\beta - 2 + 2e^{-\beta} = \frac{\psi_2(x) - 0}{x}$$

$\Rightarrow \psi_2(x) = 2x(\psi_1(\beta - 1))$ has one solution

Option (C) is correct

Q.51 (D)

(A) $\psi_1(x) = e^{-x} + x$, $x \ge 0$

$\psi_1'(x) = 1 - e^{-x} > 0 \Rightarrow \psi_1(x)$ is $\uparrow$

$\psi_1(x) \ge \psi_1(0) \quad \forall x \ge 0 \Rightarrow \psi_1(x) \ge 1$

(B) $\psi_2(x) = x^2 - 2x + 2 - 2e^{-x} \qquad x \ge 0$

$\psi_2'(x) = 2x - 2 + 2e^{-x} = 2\psi_1(x) - 2 \ge 0 \qquad \forall x \ge 0$

$\Rightarrow \psi_2(x)$ is $\uparrow$ Þ $\psi_2(x) \ge \psi_2(0) \Rightarrow \psi_2(x) \ge 0$

(C) $f(x) = 2\int_0^x (t - t^2)e^{-t^2}dt \qquad$ & $x \in \left(0, \frac{1}{2}\right)$

$= \int_0^x 2te^{-t^2}dt - \int_0^x 2t^2e^{-t^2}dl$

$= -e^{-x^2}\Big|_0^x -$

Let H(x) = $f(x) - 1 + e^{-x^2} + \frac{2}{3}x^3 - \frac{2}{5}x^5, x \in \left(0, \frac{1}{2}\right)$

H (0) = 0

$H'(x) = 2(x - x^2)e^{-x^2} - 2xe^{-x^2} + 2x^2 - 2x^4$

$= -2x^2e^{-x^2} + 2x^2 - 2x^4$

$= 2x^2\left(1 - x^2 - e^{-x^2}\right)$

$\because e^{-x} \ge 1 - x \quad \forall x \ge 0$

$\Rightarrow H'(x) \le 0$

$\Rightarrow$ H(x) is $\downarrow \qquad \Rightarrow 1 - 1(x) < 0 \forall x \in \left(0, \frac{1}{2}\right)$

Let P (x) = $g(x) - \frac{2}{3}x^3 + \frac{2}{5}x^5 - \frac{1}{7}x^7 x \in \left(0, \frac{1}{2}\right)$

$P'(x) = 2x^2e^{-x^2} - 2x^2 + 2x^4 - x^6$

$= 2x^2\left(1 - \frac{x^2}{1} + \frac{x^4}{2} - \frac{x^6}{3} + \ldots\right) - 2x^2 + 2x^4 - x^6$

$= -\frac{x^8}{3} + \frac{x^{10}}{12}$......

$\Rightarrow P'(x) \le 0$

$\Rightarrow$P(x)is $\downarrow$

$\Rightarrow P(x) \le 0$

option (D) is correct

Q.52 [182]

Let $f(x) = \left(\frac{10x}{x+1}\right)$

So, $f'(x) = 10\left(\dfrac{(x+1)-x}{(x+1)^2}\right) = \dfrac{10}{(x+1)^2} > 0 \, \forall x \in [0,10]$

So, $f(x)$ is an increasing function

So, range of $f(x)$ is $\left[0, \sqrt{\dfrac{100}{11}}\right]$

$$I = \int_0^{1/9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{1/9}^{2/3}\left[\sqrt{\frac{10x}{x+1}}\right]dx$$

$$+ \int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{9}^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$$

$$= 0 + \int_{1/9}^{2/3} dx + 2\int_{2/3}^{9} dx + 3\int_{9}^{10} dx$$

$$= \frac{2}{3} - \frac{1}{9} + 2\left(9 - \frac{2}{3}\right) + 3(10-9)$$

$$= \frac{6-1}{9} + 2 \times \frac{25}{3} + 3 = \frac{5}{9} + \frac{50}{3} + 3$$

$$= \frac{5+150+27}{9} = \frac{182}{9} = 182$$

Q.53 (A,B)

$$f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$$

$$f'(x) = \frac{(x^2 + 2x + 4)(2x - 3) - (x^2 - 3x - 6)(2x + 2)}{(x^2 + 2x + 4)^2}$$

$$f'(x) = \frac{5x(x + 4)}{(x^2 + 2x + 4)^2}$$

$f'(x)$: + (before -4), − (between -4 and 0), + (after 0)

$f(-4) = \dfrac{11}{6}$, $\quad f(0) = -\dfrac{3}{2}$, $\quad \lim\limits_{x \to \pm 0} f(x) = 1$

Range : $\left[-\dfrac{3}{2}, \dfrac{11}{6}\right]$, clearly f(x) is into

Indefinite Integration

EXERCISES

ELEMENTRY

Q.1 (2)

$$\int \left(x+\frac{1}{x}\right)^3 dx = \int \left(x^3+\frac{1}{x^3}+3x+\frac{3}{x}\right)dx$$

$$=\frac{x^4}{4}-\frac{1}{2x^2}+\frac{3x^2}{2}+3\log x + c$$

$$=\frac{x^4}{4}+\frac{3x^2}{2}+3\log x-\frac{1}{2x^2}+c.$$

Q.2 (4)

$$\int \sqrt{1+\sin x}\, dx = \int \sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}dx$$

$$=\int \sin\frac{x}{2}dx+\int \cos\frac{x}{2}dx = -2\cos\frac{x}{2}+2\sin\frac{x}{2}+c$$

$$=-2\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)+c=-2\sqrt{(1-\sin x)}+c.$$

Q.3 (2)

$$\int \frac{dx}{1+e^x}=\int \frac{e^{-x}}{1+e^{-x}}dx$$

Put $1+e^{-x}=t \Rightarrow e^{-x}dx=-dt$, then it reduces to

$$-\int\frac{dt}{t}=-\log t=-\log(1+e^{-x})$$

Q.4 (1)

$$\int \frac{dx}{x+x\log x}=\int \frac{dx}{x(1+\log x)}$$

Now putting $1+\log x = t \Rightarrow \frac{1}{x}dx = dt,$ it reduces to

$$\int \frac{dt}{t}=\log(t)=\log(1+\log x)$$

Q.5 (3)

Putting $t=\tan^{-1}x \Rightarrow dt=\frac{1}{1+x^2}dx,$ we get

$$\int \frac{e^{\tan^{-1}x}}{1+x^2}dx=\int e^t dt = e^t+c = e^{\tan^{-1}x}+c.$$

Q.6 (2)

$$\int \frac{1}{\cos^2 x(1-\tan x)^2}dx=\int\frac{\sec^2 x\,dx}{(\tan x-1)^2}$$

Put $\tan x - 1 = t \Rightarrow \sec^2 x\,dx = dt,$ then it reduces to

$$\int \frac{1}{t^2}dt=\frac{-1}{\tan x-1}+c=\frac{1}{1-\tan x}+c.$$

Q.7 (2)

Put $\sqrt{x}=t \Rightarrow \frac{1}{\sqrt{x}}dx=2dt,$ then it reduces to

$$2\int \sin t\,dt=-2\cos t+c=-2\cos\sqrt{x}+c.$$

Q.8 (1)

Put $t=\tan x \Rightarrow dt=\sec^2 x\,dx,$ then

$$\int \frac{\sec^2 x\,dx}{\sqrt{\tan^2 x+4}}=\int\frac{1}{\sqrt{t^2+2^2}}dt$$

$$=\log[\tan x+\sqrt{\tan^2 x+4}]+c.$$

Q.9 (1)

$$\int x.\tan^{-1}x\,dx=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2+1-1}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+c$$

$$=\frac{1}{2}\tan^{-1}x.(x^2+1)-\frac{1}{2}x+c$$

Q.10 (4)

Let $I=\int e^{-2x}\sin 3x\,dx$

$$=-\frac{e^{-2x}\cos 3x}{3}-\int\frac{2e^{-2x}\cos 3x}{3}dx$$

$$=-\frac{e^{-2x}\cos 3x}{3}-\frac{2}{3}\left[\frac{e^{-2x}\sin 3x}{3}+\int\frac{2e^{-2x}\sin 3x}{3}dx\right]$$

$$\Rightarrow I = -\frac{e^{-2x}\cos 3x}{3} - \frac{2e^{-2x}\sin 3x}{9} - \frac{4}{9}I$$

$$\Rightarrow \frac{13}{9}I = -e^{-2x}\left[\frac{3\cos 3x + 2\sin 3x}{9}\right]$$

Hence $I = -\frac{1}{13}e^{-2x}[3\cos 3x + 2\sin 3x]$.

Note : Students should remember this question as a formula.

Q.11 (4)

$$\int e^{2x}(-\sin x + 2\cos x)\,dx$$

$$= -\int e^{2x}\sin x\,dx + 2\int e^{2x}\cos x\,dx$$

$$= e^{2x}\cos x - 2\int e^{2x}\cos x\,dx + 2\int e^{2x}\cos x\,dx + c$$

$$= e^{2x}\cos x + c.$$

Aliter : $\int e^{2x}(2\cos x - \sin x)\,dx = e^{2x}\cos x + c$

$$\left\{\because \int e^{kx}\{k f(x) + f'(x)\}dx = e^{kx}f(x) + c\right\}$$

Q.12 (1)

$$\int \frac{\log x}{(x+1)^2}dx = \int \log x\,(x+1)^{-2}dx$$

$$= \log x.\left\{-(x+1)^{-1}\right\} - \int \frac{1}{x}.\{-(x+1)^{-1}\}dx$$

$$= \frac{-\log x}{(x+1)} + \int \frac{1}{x(x+1)}dx = \frac{-\log x}{(x+1)} + \int\left[\frac{1}{x} - \frac{1}{x+1}\right]dx$$

$$= \frac{-\log x}{x+1} + \log x - \log(x+1)$$

Q.13 (3)

$$I = \int e^x(1 - \cot x + \cot^2 x)\,dx = \int e^x(-\cot x + \mathrm{cosec}^2 x)\,dx$$

$$= e^x(-\cot x) + c = -e^x\cot x + c$$

Q.14 (1)

Putting $\tan^{-1}x = t$ and $\frac{dx}{1+x^2} = dt,$ we get

$$\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx = \int e^t(\tan t + \sec^2 t)\,dt$$

$$= e^t\tan t + c = x\,e^{\tan^{-1}x} + c$$

$$\left[\text{Using} \int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C\right].$$

Q.15 (1)

$$\int \frac{dx}{(x - x^2)} = \int\left(\frac{1}{x} + \frac{1}{1-x}\right)dx = \log x - \log(1-x) + c.$$

Q.16 (1)

$$\int \frac{dx}{1 + x + x^2 + x^3} = \int \frac{dx}{(1+x)(1+x^2)}$$

$$= \frac{1}{2}\int \frac{1}{1+x^2}dx + \frac{1}{2}\int \frac{1}{1+x}dx - \frac{1}{2}\int \frac{x}{1+x^2}dx$$

$$= \frac{1}{2}\tan^{-1}x + \log\sqrt{1+x} - \frac{1}{2}\log\sqrt{1+x^2} + c.$$

Q.17 (1)

$$\int \frac{x}{(x-2)(x-1)}dx = -\int \frac{1}{x-1}dx + \int \frac{2}{x-2}dx$$

$$= -\log_e(x-1) + 2\log_e(x-2) + c = \log_e\frac{(x-2)^2}{(x-1)} + c.$$

Q.18 (1) We have

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$$

$$\Rightarrow 1 = A(x^2+1) + (Bx+C)(x-1)$$

If $x = 1$, then $A = \frac{1}{2}$ .....(i)

$A - C = 1 \Rightarrow C = -\frac{1}{2}$ .....(ii)

$A + B = 0 \Rightarrow B = -\frac{1}{2}$ .....(iii)

Putting these values, we get

$$\frac{1}{(x-1)(x^2+1)} = \frac{1}{2}.\frac{1}{(x-1)} - \frac{x+1}{2(x^2+1)}$$

Hence

$$\int \frac{1}{(x-1)(x^2+1)}dx = \frac{1}{2}\int \frac{dx}{(x-1)} - \frac{1}{2}\int \frac{x+1}{x^2+1}dx$$

$$= \frac{1}{2}\log(x-1) - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + c.$$

Q.19 (2)

$$\int \frac{x^2+x-1}{x^2+x-6}dx = \int \left[1+\frac{5}{x^2+x-6}\right]dx$$

$$= \int \left[1+\frac{5}{(x+3)(x-2)}\right]dx = \int dx + \int \frac{dx}{x-2} - \int \frac{dx}{x+3}$$

$$= x + \log(x-2) - \log(x+3) + c.$$

Q.20 (3)

$$\int \frac{dx}{(x^2+1)(x^2+4)} = \frac{1}{3}\left[\int \frac{dx}{x^2+1} - \int \frac{dx}{x^2+4}\right]$$

$$= \frac{1}{3}\left[\tan^{-1}x - \frac{1}{2}\tan^{-1}\frac{x}{2}\right] + c = \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + c.$$

Q.21 (4)

$$\int \frac{1}{x-x^3}dx = \int \frac{1}{x(1+x)(1-x)}dx$$

$$= \frac{1}{2}\int \left(\frac{2}{x} - \frac{1}{1+x} + \frac{1}{1-x}\right)dx$$

$$= \frac{1}{2}[2\log x - \log(1+x) - \log(1-x)] = \frac{1}{2}\log\frac{x^2}{(1-x^2)} + c.$$

Q.22 (2)

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}dx$$

$$= \frac{1}{(a-b)}\int (x+a)^{1/2}dx - \frac{1}{(a-b)}\int (x+b)^{1/2}dx$$

$$= \frac{2}{3(a-b)}[(x+a)^{3/2} - (x+b)^{3/2}] + c.$$

Q.23 (4)

$$\int \frac{1}{\cos x(1+\cos x)}dx = \int \frac{dx}{\cos x} - \int \frac{dx}{1+\cos x}$$

$$= \int \sec x\, dx - \frac{1}{2}\int \sec^2\frac{x}{2}dx$$

$$= \log(\sec x + \tan x) - \tan\frac{x}{2} + c.$$

Q.24 (1)

$$\int \sin^3 x \cos^2 x\, dx = \int (1-\cos^2 x)\cos^2 x . \sin x\, dx$$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$, then it reduces to

$$-\int (t^2 - t^4)dt = \frac{t^5}{5} - \frac{t^3}{3} + c = \frac{(\cos x)^5}{5} - \frac{(\cos x)^3}{3} + c.$$

Q.25 (2)

$$\int \sin 2x \cos 3x\, dx = \frac{1}{2}\int 2(\sin 2x \cos 3x)\, dx$$

$$= \frac{1}{2}\int (\sin 5x - \sin x)\, dx = \frac{1}{2}\left[-\frac{\cos 5x}{5} + \cos x\right] + c$$

$$= \frac{1}{2}\left[\cos x - \frac{\cos 5x}{5}\right] + c.$$

Q.26 (2)

$$\int \frac{e^x}{(1+e^x)(2+e^x)}dx = \int \left\{\frac{e^x}{1+e^x} - \frac{e^x}{2+e^x}\right\}dx$$

Now put $1+e^x = t$ and $2+e^x = t$, then the required integral

$$= \log(1+e^x) - \log(2+e^x) = \log\left(\frac{1+e^x}{2+e^x}\right) + c.$$

Q.27 (3)

$$\int \frac{dx}{5+4\cos x}$$

$$= \int \frac{dx}{5+4\left[\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]} = \int \frac{\sec^2\frac{x}{2}}{9+\tan^2\frac{x}{2}}dx$$

Put $\tan\frac{x}{2} = t$, then it reduces to

$$2\int \frac{dt}{3^2+t^2} = \frac{2}{3}\tan^{-1}\left[\frac{1}{3}\tan\frac{x}{2}\right] + c$$

Aliter : Apply direct formula

i.e., $\int \frac{1}{a+b\cos x}dx$, $\{a > b\}$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right] + c$$

We get $\int \frac{dx}{5+4\cos x} = \frac{2}{3}\tan^{-1}\left\{\frac{1}{3}\tan\frac{x}{2}\right\} + c.$

Q.28 (2)

Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt \Rightarrow dx = 2t\,dt,$ then

$$\int \sin\sqrt{x}\,dx = 2\int t\sin t\,dt = 2(-t\cos t + \sin t) + c$$

$$= 2(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}) + c.$$

Trick : Let $\sec 2x = t,$ then $\sec 2x\tan 2x\,dx = \frac{1}{2}dt$

$$\therefore \frac{1}{2}\int (t^2-1)dt = \frac{1}{6}t^3 - \frac{1}{2}t + c = \frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + c.$$

Q.29 (4)

$$\int \frac{x^2+1}{x^4+1}dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}dx = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+2}$$

Put $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt,$ then the required integral is $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + c.$

Q.30 (4)

The given function can be written as

$$\int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-1}dx$$

Put $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt,$ then it reduces to

$$\int \frac{dt}{t^2-1} = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + c$$

$$= \frac{1}{2}\log\left(\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right) + c = \frac{1}{2}\log\left(\frac{x^2-x+1}{x^2+x+1}\right) + c.$$

JEE-MAIN
OBJECTIVE QUESTIONS

Q.1 (1)

$$\int \frac{dx}{\sin x.\sin(x+\alpha)}$$

$$= \frac{1}{\sin\alpha}\int \frac{\sin(\alpha+x-x)}{\sin x\,\sin(x+\alpha)}dx$$

$$= \text{cosec }\alpha \int \frac{\sin(x+\alpha)\cos x - \cos(x+\alpha)\sin x}{\sin x\sin(x+\alpha)}$$

$$= \text{cosec }\alpha \left[\int \cot x\,dx - \int \cot(x+\alpha)\right] + C$$

$= \text{cosec }\alpha\ [\log|\sin x| - \log|\sin(x+\alpha)|] + C$

$$= \text{cosec }\alpha \log\left|\frac{\sin x}{\sin(x+\alpha)}\right| + C$$

Q.2 (1)

$$I = \int \frac{1}{1+\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$= \int \frac{1}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}dx = \frac{1}{2}\int \sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

$$= \tan\left(\frac{x}{2}-\frac{\pi}{4}\right) + b \Rightarrow a = -\frac{\pi}{4},\ b \in R$$

Q.3 (2)

$$I = -\frac{1}{2}\cos 2x - \frac{\sin 2x}{2} + b = -\frac{1}{\sqrt{2}}\sin\left(2x+\frac{\pi}{4}\right) + b$$

$$= \frac{1}{\sqrt{2}}\sin\left(2x+\frac{5\pi}{4}\right) + b \quad \therefore\ a = -\frac{5\pi}{4},\ b \in R$$

Q.4 (1)

$I = \int \frac{\cos 2x}{\cos x}dx = \int \frac{2\cos^2 x - 1}{\cos x}dx = 2\sin x - \int \sec x\,dx = 2\sin x - \ell n|\sec x + \tan x| + C$

Q.5 (3)

Sol. $\int (f(x)\,g''(x) - f''(x)\,g(x))\,dx$

$= f(x)\int g''(x)\,.\,dx - \int f'(x)\,g'(x)\,dx - g(x)\int f''(x)\,dx + \int g'(x)\,f'(x)\,dx$

$= f(x)\,g'(x) - f'(x)\,g(x) + c$

Q.6 (2)

$$\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}\,.\quad (4)x$$

$$\int \frac{(\sin^4 + \cos^4 x)(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{(\sin^4 x + \cos^4 x)}\,dx$$

$$= -\int \cos 2x\,dx = -\frac{\sin 2x}{2} + c$$

Q.7 (4)

$$\int \frac{\tan^{-1}x - \cot^{-1}x}{\tan^{-1}x + \cot^{-1}x}\,dx$$

$$\Rightarrow \int \frac{\tan^{-1}x - \cot^{-1}x}{\pi/2}\,dx$$

$$\Rightarrow \frac{2}{\pi}\int \left(\tan^{-1}x - \frac{\pi}{2} + \tan^{-1}x\right)dx$$

$$\Rightarrow \frac{2}{\pi}\int \left(2\tan^{-1}x - \frac{\pi}{2}\right)dx$$

$$\Rightarrow \frac{4}{\pi}\left[x\tan^{-1}x - \frac{1}{2}\ell n\,|1+x^2|\right] - x + c$$

$$\Rightarrow \frac{4}{\pi}\,x\tan^{-1}x - \frac{2}{\pi}\,\ell n\,|1+x^2| - x + c$$

Q.8 (2)

$$\int \frac{a^{\sqrt{x}}}{\sqrt{x}}\,dx$$

put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}\,dx = dt$ Þ $\frac{dx}{\sqrt{x}} = 2dt$

$$= 2\int a^t\,dt = \frac{2a^t}{\ell n\,a} + c = 2\,\frac{a^{\sqrt{x}}}{\ell n\,a} + c$$

Q.9 (3)

$I = \int 5^{5^{5^x}}.5^{5^x}.5^x\,dx$ Let $5^{5^{5^x}} = t$

$$5^{5^{5^x}}.\ell n5.5^{5^x}\,\ell n5.5^x\,\ell n\,5\,dx = dt$$

$$5^{5^{5^x}}.5^{5^x}.5^x\,dx = \frac{dt}{(\ell n\,5)^3}$$

$$I = \int \frac{dt}{(\ell n\,5)^3} = \frac{t}{(\ell n\,5)^3} + c = \frac{5^{5^{5^x}}}{(\ell n\,5)^3} + c$$

Q.10 (1)

$$\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx$$

$$\int \frac{\sqrt{\tan x}\,\sec^2 x}{\tan x}\,dx$$

$\tan x = t^2 \Rightarrow \sec^2 x\,dx = 2t\,dt$

$$\int \frac{t.2t\,dt}{t^2} = 2t + c = 2\sqrt{\tan x} + c$$

Q.11 (4)

$$\int \frac{2^x}{\sqrt{1-4^x}}\,dx$$

$2^x = t \Rightarrow 2^x \ln 2\,dx = dt \Rightarrow 2^x\,dx = \frac{dt}{\ell n\,2}$

$$\frac{1}{\ell n\,2}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\ell n\,2}\sin^{-1}(2^x) + c$$

Q.12 (3)

$$\int \tan^3 2x \sec 2x\,dx$$

$$\int \tan 2x\,(\sec^2 2x - 1)\sec 2x\,dx$$

$$= \int \frac{\sin 2x}{\cos^4 2x}\,dx - \int \frac{\sin 2x}{\cos^2 2x}\,dx$$

put $\cos 2x = t$

$\sin 2x\,dx = -\frac{dt}{2}$

$$= -\frac{1}{2}\int \frac{dt}{t^4} + \frac{1}{2}\int \frac{dt}{t^2}$$

$$= -\frac{1}{2}\left[\frac{t^{-3}}{-3}\right] - \frac{1}{2}\,\frac{1}{t} + c$$

$$= \frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + c$$

Q.13 (1)

$$\int \sqrt{\frac{e^x - 1}{e^x + 1}}\,dx = \int \frac{e^x - 1}{\sqrt{e^{2x} - 1}}\,dx$$

$$= \int \frac{e^x}{\sqrt{e^{2x}-1}}\,dx - \int \frac{dx}{\sqrt{e^{2x}-1}}$$

$$= \int \frac{dt}{\sqrt{t^2-1}} - \int \frac{e^x}{e^x\sqrt{e^{2x}-1}}\,dx$$

$= \int \frac{dt}{\sqrt{t^2-1}} - \int \frac{du}{u\sqrt{u^2-1}}$

$= \ell n\left(e^x + \sqrt{e^{2x}-1}\right) - \sec^{-1}(e^x) + c$

Q.14 (3)

$I = \int \sqrt{\frac{1-\cos x}{\cos x}}\,dx = \int \sqrt{\frac{2\sin^2\frac{x}{2}}{\cos x}}\,dx = \int \frac{\sqrt{2}\sin\frac{x}{2}}{\sqrt{2\cos^2\frac{x}{2}-1}}\,dx$

Let $\cos\frac{x}{2} = t \Rightarrow -\frac{1}{2}\sin\frac{x}{2}\,dx = dt$

$I = -2\sqrt{2}\int \frac{dt}{\sqrt{2t^2-1}} = \frac{-2\sqrt{2}}{\sqrt{2}}\int \frac{dt}{\sqrt{t^2-\frac{1}{2}}}$

$= -2\int \frac{dt}{\sqrt{t-\left(\frac{1}{\sqrt{2}}\right)^2}}$

$\Rightarrow I = -2\log\left|t+\sqrt{t^2-\frac{1}{2}}\right| + c$

$= -2\log\left|\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right| + c$

Q.15 (1)

$I = \int \frac{dx}{x\sqrt{1-x^3}} = \int \frac{x^2\,dx}{x^3\sqrt{1-x^3}}$ Put $1-x^3 = t^2$

$\Rightarrow x^2\,dx = \frac{-2t\,dt}{3}$

$\Rightarrow I = -\frac{2}{3}\int \frac{t.\,dt}{(1-t^2).t}$

$= \frac{2}{3}\int \frac{dt}{t^2-1}$

$= \frac{2}{3}\times\frac{1}{2}\,\ell n\left|\frac{t-1}{t+1}\right| + C$

$= \frac{1}{3}\,\ell n\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + C$

Q.16 (1)

$\int \frac{\sin^2 x}{1+\sin^2 x}\,dx = \int \frac{\sin^2 x+1-1}{1+\sin^2 x}\,dx$

$= \int dx - \int \frac{dx}{1+\sin^2 x}$

$= \int dx - \int \frac{\sec^2 x}{\sec^2 x+\tan^2 x}\,dx$

$= \int dx - \int \frac{\sec^2 x}{1+2\tan^2 x}\,dx$

$= \int dx - \int \frac{dt}{1+2t^2} = x - \left(\frac{1}{\sqrt{2}}\right)\tan^{-1}\sqrt{2}\,t + c$

$= x - \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + c$

Q.17 (1)

$\int \left\{\frac{(\log x-1)}{1+(\log x)^2}\right\}^2 dx$

$\ell n\, x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$= \int e^t\left(\frac{t-1}{t^2+1}\right)^2 dt = \int e^t\left(\frac{t^2+1-2t}{(t^2+1)^2}\right)dt$

$= \int e^t\left\{\underset{f(t)}{\underset{\uparrow}{\frac{1}{(t^2+1)}}} - \underset{f'(t)}{\underset{\uparrow}{\frac{2t}{(1+t^2)^2}}}\right\}dt$

$= \frac{e^t}{1+t^2} = \frac{x}{1+\log^2 x} + c$

Q.18 (2)

$\int \frac{\sin x}{\sin(x-a)}\,dx$

Let $x - a = t \Rightarrow dx = dt$

$\int \frac{\sin(a+t)}{\sin t}\,dt$

$= \int \frac{\sin a\cos t+\cos a\sin t}{\sin t}\,dt$

$= \sin a\int \cot t\,dt + \cos a\int dt$

$= t \,.\, \cos a + \sin a \,.\, \ell n(\sin t) + c$
$= (x - a)\cos a + \sin a\,\ell n\sin(x - a) + c$
$= x\cos a + \sin a\,\ell n\sin(x - a) + c$
$(\cos a, \sin a)$

Q.19 (1)

$\int \frac{\ell n|x|}{x\sqrt{1+\ell n\, x}}\,dx$

$1 + \ell n\, x = t^2 \Rightarrow \frac{1}{x} dx = 2t\, dt$

$\int \frac{(t^2-1)\, 2t\, dt}{t} = 2\int (t^2-1)\, dt$

$= 2\left[\frac{t^3}{3} - t\right] + c = \frac{2}{3} t\, [t^2 - 3] + c = \frac{2}{3}\sqrt{1+\ell n\, x}\ [1$

$+ \ell n\, x - 3] + c = \frac{2}{3}\sqrt{1+\ell n\, x}\ [\ell n\, x - 2] + c$

Q.20 (3)

$\int \{1 + 2\tan x\, (\tan x + \sec x)\}^{1/2} dx$

$\int \{1 + 2\tan^2 x + 2\tan x \sec x\}^{1/2} dx$

$\int \{\sec^2 x - \tan^2 x + 2\tan^2 x + 2\tan x \sec x\}^{1/2} dx$

$= \int (\sec x + \tan x)\, dx$

$= \ell n \sec x + \ell n\, (\sec x + \tan x) + c$

$= \ell n \sec x\, (\sec x + \tan x) + c$

Q.21 (3)

$\int (x-1)\, e^{-x}\, dx$

$= \int x\, e^{-x}\, dx - \int e^{-x}\, dx$

$= -xe^{-x} + \int e^{-x}\, dx - \int e^{-x}\, dx$

$= -xe^{-x} + c$

Q.22 (1)

$\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right) dx$

$\tan^{-1} x = t \Rightarrow \frac{dx}{1+x^2} = dt$

$\int e^t\, (1 + \tan t + \tan^2 t)\, dt$

$= \int e^t\, (\sec^2 t + \tan t)\, dt$

$= e^t \tan t + c$

$= e^{\tan^{-1}x} \tan(\tan^{-1}x) + c = x\, e^{\tan^{-1}x} + c$

Q.23 (3)

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}(x + \sqrt{x})\, dx$

$\sqrt{x} = t \Rightarrow \frac{1}{\sqrt{x}} dx = 2\, dt$

$2\int e^t\, (t^2 + t)\, dt$

$= 2\int e^t\, (t^2 + 2t) - 2\int e^t\, t\, dt$

$= 2\, e^t\, (t^2) - 2[t\, e^t - e^t] + c$

$= 2e^{\sqrt{x}}.x - 2[\sqrt{x}\, e^{\sqrt{x}} - e^{\sqrt{x}}] + c$

$= 2e^{\sqrt{x}}[x - \sqrt{x} + 1] + c$

Q.24 (4)

$\int e^{\tan\theta}(\sec\theta - \sin\theta)\, d\theta$

$\tan\theta = t \Rightarrow d\theta = \frac{dt}{1+t^2}$

$I = \int \frac{e^t}{1+t^2}\left(\sqrt{1+t^2} - \frac{t}{\sqrt{1+t^2}}\right) dt$

$= \int e^t\left(\frac{1}{\sqrt{1+t^2}} - \frac{t}{\sqrt{(t^2+1)^3}}\right) dt$

$= e^t\frac{1}{\sqrt{(t^2+1)}} + c = e^{\tan\theta}\frac{1}{\sec\theta} + c$

$= e^{\tan\theta}\cos\theta + c$

Q.25 (2)

$y = \int \frac{dx}{x^2+x+1} = \int \frac{dx}{x^2 + x + \frac{1}{4} + 1 - \frac{1}{4}}$

$= \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

Q.26 (3)

$I = \int e^{3x}\cos 4x\,dx$

$= e^{3x}(A\sin 4x + B\cos 4x) + C$ ....(i)

$I = \frac{1}{4}e^{3x}\sin 4x - \int \frac{3}{4}e^{3x}\sin 4x\,dx$

$= \frac{1}{4}e^{3x}\sin 4x + \frac{3}{16}e^{3x}\cos 4x - \int \frac{9}{16}e^{3x}\cos 4x\,dx$

$\frac{25}{16}I = \frac{1}{16}\left(4e^{3x}\sin 4x + 3e^{3x}\cos 4x\right)$

comparing with equation (i)

$\Rightarrow A = \frac{4}{25},\ B = \frac{3}{25}$

$\Rightarrow \frac{A}{B} = \frac{4}{3} \Rightarrow 3A = 4B$

Q.27 (3)

$\int \frac{dx}{x^3(1+x)} = \frac{A}{x^2} + \frac{B}{x} + \ell n\left(\frac{x}{x+1}\right) + c$

$\frac{1}{x^3(1+x)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \frac{d}{(x+1)}$

$1 = ax^2(x+1) + bx(x+1) + c(x+1) + dx^3$

put $x = 0 \Rightarrow c = 1$

put $x = -1 \Rightarrow d = -1$

put $x = 1$

$1 = 2a + 2b + 2c + d$

$1 = 2a + 2b + 2 - 1 \Rightarrow a + b = 0$

put $x = 2$

$1 = 12a + 6b + 3c + 8d$

$1 = 12a + 6b + 3 - 8$

$12a + 6b = 6 \Rightarrow 2a + b = 1 \Rightarrow a = 1$

$\int \frac{dx}{x^3(1+x)} = \int\left(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x+1}\right)dx$

$= \ell n\,x + \frac{1}{x} - \frac{1}{2x^2} - \ell n\,(x+1) + c$

$= -\frac{1}{2x^2} + \frac{1}{x} + \ell n\left(\frac{x}{x+1}\right) + c$

$A = -1/2,\ B = 1$

Aliter : $\int \frac{1+x^3-x^3}{x^3(1+x)} = \int \frac{1+x^3}{x^3(1+x)} - \int \frac{dx}{1+x}$

Q.28 (2)

$\int \frac{x^3-1}{x(x^2+1)}dx = \int \frac{x^2}{x^2+1}dx - \int \frac{1}{x(x^2+1)}dx$

$= \int dx - \int \frac{dx}{x^2+1} - \int \frac{1}{x^3(1+x^{-2})}dx$

Let $1 + x^{-2} = t \Rightarrow \frac{dx}{x^3} = -\frac{dt}{2}$

$= x - \tan^{-1}x + \frac{1}{2}\int \frac{dt}{t}$

$= x - \tan^{-1}x + \frac{1}{2}\,\ell n\,|1 + x^{-2}| + c$

$= x - \tan^{-1}x + \frac{1}{2}\,\ell n\,(x^2+1) - \ell n\,x + c$

Q.29 (4)

$y = \int \frac{dx}{x^3\left(1+\frac{1}{x^2}\right)^{3/2}}$ put $1 + \frac{1}{x^2} = t^2 \Rightarrow -\frac{2}{x^3}$

$dx = 2t\,dt$

$\Rightarrow y = \int \frac{-t\,dt}{t^3} = -\int \frac{dt}{t^2} = \frac{1}{t} + C = \frac{x}{\sqrt{1+x^2}} + C$

$\because y(0) = 0 \Rightarrow C = 0$

$\therefore y(1) = \frac{1}{\sqrt{2}}$

Q.30 (2)

$I = \int \frac{\cos 2x\,dx}{(\sin x + \cos x)^2} = \int \frac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)^2}dx$

$= \int \frac{\cos x - \sin x}{\sin x + \cos x}dx$ Put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\,dx = dt$

$\Rightarrow I = \int \frac{dt}{t} = \ell n\,|t| + C = \ell n|\sin x + \cos x| + C$

Q.31 (3)

$\int (1 + \tan x\tan(x+\alpha))dx$

$= \int \frac{\sin x\sin(x+\alpha) + \cos x\cos(x+\alpha)}{\cos x\cos(x+\alpha)}$

$= \int \frac{\cos(x+\alpha-x)}{\cos x\cos(x+\alpha)}dx$

$= \cot\alpha \int \frac{\sin(x+\alpha-x)}{\cos x\cos(x+\alpha)}dx$

$= \cot\alpha \left[\int \frac{\sin(x+\alpha)\cos x}{\cos x\cos(x+\alpha)} - \frac{\cos(x+\alpha)\sin x}{\cos x\cos(x+\alpha)}\, dx\right]$

$= \cot\alpha \left[\int \tan(x+\alpha)\, dx - \int \tan x\, dx\right]$

$= \cot\alpha\ [\ell n\,|\sec(x+\alpha)| - \ell n\,|\sec x|]$

$= \cot\alpha\ \ell n \left|\frac{\cos x}{\cos(x+\alpha)}\right| + C$

$= \cot\alpha\ \ell n \left(\left|\frac{\sec(x+\alpha)}{\sec x}\right|\right) + C$

Q.32 (2)

$\int \frac{dx}{\cos^3 x\sqrt{\sin 2x}}$

$\int \frac{dx}{\cos^3 x\sqrt{2\sin x\cos x}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\cos^4 x\sqrt{\tan x}} = \frac{1}{\sqrt{2}} \int \frac{\sec^4 x}{\sqrt{\tan x}}\, dx$

$= \frac{1}{\sqrt{2}} \int \frac{(1+\tan^2 x)}{\sqrt{\tan x}} \sec^2 x\, dx$

put $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$

$= \frac{1}{\sqrt{2}} \int \left(\frac{1+t^4}{t}\right) 2t\, dt = \sqrt{2} \int (1+t^4)\, dt$

$= \sqrt{2}\left[t + \frac{t^5}{5}\right] + c$

$= \sqrt{2}\left[\tan^{1/2} x + \frac{\tan^{5/2} x}{5}\right] + c$

Q.33 (2)

$\int \frac{\cos 4x + 1}{\cot x - \tan x} = A\cos 4x + B$

$I = \int \frac{(\cos 4x + 1)}{(\cos^2 x - \sin^2 x)} \cos x \sin x\, dx$

$= \int \left(\frac{2\cos^2 2x}{\cos 2x}\right) (\cos x \sin x)\, dx$

$= \int \cos 2x \sin 2x\, dx$

$= \frac{1}{2} \int \sin 4x\, dx = -\frac{\cos 4x}{8} + B$

Q.34 (2)

$I = \int \sqrt{\frac{1}{(\tan x)^{11}(\cos^8 x)}}\, dx$

$= \int (\tan x)^{-11/2} \sec^4 x dx$

$= \int (\tan x)^{-11/2}(1+\tan^2 x)\sec^2 x dx$ put $\tan x = t$

$\Rightarrow \sec^2 x dx = dt$

$\therefore\ I = \int (t^{-11/2} + t^{-7/2})\, dt$

$= \frac{-2}{9} t^{-9/2} + \frac{-2}{5} t^{-5/2} + C$

$= \frac{-2}{9}(\tan x)^{-9/2} + \frac{-2}{5}(\tan x)^{-5/2} + C \Rightarrow A = \frac{1}{9}$, B

$= \frac{1}{5}$

Q.35 (2)

$\int 4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}\, dx$

$2\int \left(2\sin x\cos\frac{x}{2}\right)\cos\frac{3x}{2}\, dx$

$2\int \left(\sin\frac{3x}{2} + \sin\frac{x}{2}\right)\cos\frac{3x}{2}\, dx$

$= \int 2\sin\frac{3x}{2}\cos\frac{3x}{2}\, dx + \int 2\sin\frac{x}{2}\cos\frac{3x}{2}\, dx$

$= \int \sin 3x\, dx + \int ((\sin 2x) - \sin x)\, dx$

$= -\frac{\cos 3x}{3} - \frac{\cos 2x}{2} + \cos x + c$

Q.36 (2)

$\int \sin x.\cos x.\cos 2x \cos 4x.\cos 8x.\cos 16x\, dx$

$= \frac{1}{2} \int (\sin 2x.\cos 2x) \cos 4x.\cos 8x.\cos 16x\, dx$

$= \frac{1}{4} \int (\sin 4x.\cos 4x) \cos 8x.\cos 16x\, dx$

$= \frac{1}{8} \int (\sin 8x.\cos 8x) \cos 16x\, dx$

$= \frac{1}{16} \int \sin 16x \cos 16x\, dx$

$= \frac{1}{32}\int \sin 32x\, dx = -\frac{1}{32} \times \frac{\cos 32x}{32}$

$= -\frac{1}{1024} \cos 32x + c$

Q.37 (3)

$\int \frac{dx}{\sin^6 x + \cos^6 x}$

$= \int \frac{dx}{\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x}$

$= \int \frac{\sec^4 x\, dx}{\tan^4 x + 1 - \tan^2 x}$

$= \int \frac{(1 + \tan^2 x)\sec^2 x\, dx}{\tan^4 x - \tan^2 x + 1}$

$\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$= \int \frac{(1+t^2)\, dt}{t^4 - t^2 + 1} = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 1} dt$

Let $t - \frac{1}{t} = u \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = du$

$= \int \frac{du}{1+u^2} = \tan^{-1} u + c = \tan^{-1}\left(t - \frac{1}{t}\right) + c$

$= \tan^{-1}\left(\tan x - \frac{1}{\tan x}\right) + c$

$= \tan^{-1}(\tan x - \cot x) + c$

Q.38 (1)

$\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}} = a\sqrt{\cot x} + b\sqrt{\tan^3 x} + c$

$= \int \frac{\sec^4 x\, dx}{\sqrt{\tan^3 x}}$

$\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$

$\int \frac{(1+\tan^2 x)}{\tan^{3/2} x} \sec^2 x\, dx$

$= \int \left(\frac{1+t^4}{t^3}\right) 2t\, dt = 2\int \left(\frac{1}{t^2} + t^2\right) dt = -\frac{2}{t} + \frac{2}{3} t^3$

$+ c = -2\sqrt{\cot x} + \frac{2}{3}\sqrt{\tan^3 x} + c$

Q.39 (1)

$\int \frac{dx}{\cos x - \sin x} = \int \frac{\left(1 + \tan^2 \frac{x}{2}\right)}{1 - \tan^2 \frac{x}{2} - 2\tan \frac{x}{2}}$

$= \int \frac{\sec^2 \frac{x}{2}\, dx}{1 - \tan^2 \frac{x}{2} - 2\tan \frac{x}{2}}$

Put $\tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2}\, dx = 2dt$

$= 2\int \frac{dt}{1 - t^2 - 2t} = 2\int \frac{-dt}{(t^2 + 2t - 1)}$

$= 2\int \frac{-dt}{(t^2 + 2t + 1 - 2)} = 2\int \frac{dt}{2 - (t+1)^2}$

$= \frac{2}{2\sqrt{2}}\, \ell n \frac{\sqrt{2} + t + 1}{\sqrt{2} - (t+1)} + c$

$= \frac{1}{\sqrt{2}}\, \ell n \frac{\sqrt{2} + \tan \frac{x}{2} + 1}{\sqrt{2} - \left(\tan \frac{x}{2} + 1\right)} + c$

Q.40 (1)

$\int \frac{dx}{\sqrt{\sin^3 x \cos x}} = \int \frac{dx}{\sqrt{\tan^3 x . \cos^4 x}}$

$= \int \frac{\sec^2 x\, dx}{\sqrt{\tan^3 x}}$ [$\tan x = t \Rightarrow \sec^2 x\, dx = dt$]

$= \int \frac{dt}{t^{3/2}} = \frac{t^{-3/2+1}}{1 - 3/2} + c = \frac{-2}{\sqrt{\tan x}} + c$

Q.41 (1)

$\int \frac{1}{x(x^n + 1)} dx = \int \frac{1}{x^{1+n}(1 + x^{-n})} dx$

$1 + x^{-n} = t \Rightarrow -nx^{-n-1}\, dx = dt \Rightarrow \frac{dx}{x^{n+1}} = \frac{-1}{n} dt$

$= -\frac{1}{n}\int \frac{dt}{t} = -\frac{1}{n}\, \ell n\, (1 + x^{-n}) + c$

$= -\frac{1}{n}\, \ell n\left(\frac{x^n + 1}{x^n}\right) + c = \frac{1}{n}\, \ell n\left(\frac{x^n}{1 + x^n}\right) + c$

Q.42 (3)

$$\int \frac{4e^{x}+6e^{-x}}{9e^{x}-4e^{-x}}\,dx$$

$$\frac{4e^{x}+6e^{-x}}{9e^{x}-4e^{-x}} = A + B\,\frac{9e^{2x}.2}{9e^{2x}-4}$$

$$\frac{\cdot 4e^{x}+6e^{-x}}{9e^{x}-4e^{-x}} = \frac{9Ae^{2x}-4A+18\cdot Be^{2x}}{9e^{2x}-4}$$

$$= \frac{9Ae^{x}-4Ae^{-x}+18Be^{x}}{9e^{x}-4e^{-x}}$$

$6 = -4A$ | $4 = 9A + 18B$

$A = -\frac{3}{2}$ | $4 = -\frac{27}{2} + 18B$

$18B = \frac{35}{2}$

$B = \frac{35}{36}$

Q.43 (4)

$$I = \int \frac{1}{x^{5}\left(1+\frac{1}{x^{4}}\right)^{3/4}}\,dx \quad \text{put } 1+\frac{1}{x^{4}} = t$$

$$\Rightarrow -\frac{4}{x^{5}}\,dx = dt$$

$$\Rightarrow I = -\frac{1}{4}\int \frac{dt}{t^{3/4}} = -\frac{1}{4}\cdot\frac{t^{1/4}}{1/4} + C$$

$$= -\left(1+\frac{1}{x^{4}}\right)^{1/4} + C$$

Q.44 (3)

$$\int \frac{1-x^{7}}{x(1+x^{7})}\,dx = \int \frac{1}{x(1+x^{7})}\,dx - \int \frac{x^{6}}{(1+x^{7})}\,dx$$

$$= \int \frac{1}{x^{8}(x^{-7}+1)}\,dx - \int \frac{x^{6}}{(1+x^{7})}\,dx$$

$1 + x^{-7} = t$ $1 + x^{7} = u$

$\frac{-7}{x^{8}}\,dx = dt$ $x^{6}\,dx = \frac{du}{7}$

$\frac{dx}{x^{8}} = \frac{-1}{7}\,dt$

$$= -\frac{1}{7}\int \frac{dt}{t} - \frac{1}{7}\int \frac{du}{u} = -\frac{1}{7}\,\ell n\, t - \frac{1}{7}\,\ell n\, u$$

$$= -\frac{1}{7}\,\ell n\,(1 + x^{-7}) - \frac{1}{7}\,\ell n\,(1 + x^{7}) + c$$

$$= -\frac{1}{7}\,\ell n\left(\frac{x^{7}+1}{x^{7}}\right) - \frac{1}{7}\,\ell n\,(1 + x^{7}) + c$$

$$= -\frac{2}{7}\,\ell n\,(1 + x^{7}) + \ell n\, x + c$$

Q.45 (2)

$$\int \frac{3x^{4}-1}{(x^{4}+x+1)^{2}}\,dx = \int \frac{3x^{4}-1}{x^{2}\left(x^{3}+1+\frac{1}{x}\right)^{2}}\,dx$$

$$= \int \frac{3x^{2}-1/x^{2}}{\left(x^{3}+1+\frac{1}{x}\right)^{2}}\,dx$$

$$x^{3} + 1 + \frac{1}{x} = t \Rightarrow \left(3x^{2}-\frac{1}{x^{2}}\right)dx = dt$$

$$= \int \frac{dt}{t^{2}} = -\frac{1}{t} + c$$

$$= -\frac{1}{\left(x^{3}+1+1/x\right)} + c = -\frac{x}{x^{4}+x+1} + c$$

JEE-ADVANCED
OBJECTIVE QUESTIONS

Q.1 (B)

$$\int \tan(x-\alpha)\tan(x+\alpha)\tan 2x\,dx$$

$$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\tan A\tan B = 1 - \frac{(\tan A+\tan B)}{\tan(A+B)}$$

$$\tan(x-\alpha)\tan(x+\alpha) = 1 - \frac{(\tan(x+\alpha)+\tan(x-\alpha))}{\tan 2x}$$

$\tan 2x\tan(x-\alpha)\tan(x+\alpha)$
$= \tan 2x - \tan(x+\alpha) - \tan(x-\alpha)$

$$\int \tan 2x\tan(x-\alpha)\tan(x+\alpha)\,dx$$

$$= \int \tan 2x\,dx - \int \tan(x+\alpha)\,dx - \tan(x-\alpha)dx$$

$= \frac{1}{2}\ell n \sec 2x - \ell n \sec(x + a) - \ell n \sec(x - \alpha) + c$

$= \ell n \frac{\sqrt{\sec 2x}}{\sec(x+\alpha)\sec(x-\alpha)} + c$

Q.2 (C)

$\int \sqrt{\frac{x-1}{x+1}}.\frac{1}{x^2}dx = \int \sqrt{\frac{1-1/x}{1+1/x}}.\frac{1}{x^2}dx$

$\frac{1}{x} = t \Rightarrow \frac{dx}{x^2} = -dt$

$\int -\sqrt{\frac{1-t}{1+t}}dt$

put $t = \cos 2\theta \Rightarrow dt = -2\sin 2\theta\, d\theta$

$= \int 2\frac{\sin\theta}{\cos\theta}(2\sin\theta\cos\theta)\,d\theta$

$= 2\int 2\sin^2\theta\, d\theta = 2\int(1-\cos 2\theta)\,d\theta$

$= 2\theta - \sin 2\theta + c = \cos^{-1} t - \sqrt{1-t^2} + c$

$= \cos^{-1}\frac{1}{x} - \sqrt{1-\frac{1}{x^2}} + c = \sec^{-1} x - \frac{\sqrt{x^2-1}}{x} + c$

Q.3 (B)

$\int \frac{x\,dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$

Let $1 + x^2 = t^2 \Rightarrow xdx = tdt$

$\int \frac{t\,dt}{\sqrt{t^2+t^3}} = \int \frac{dt}{\sqrt{1+t}}$

$= 2\sqrt{1+t} + c = 2\sqrt{1+\sqrt{1+x^2}} + c$

Q.4 (A)

$\int \sqrt{\frac{a+x}{a-x}} - \sqrt{\frac{a-x}{a+x}}\,dx = \int \frac{(a+x)-(a-x)}{\sqrt{a^2-x^2}}dx$

$= \int \frac{2x}{\sqrt{a^2-x^2}}dx$

$a^2 - x^2 = t^2 \Rightarrow xdx = tdt$

$= -2\int \frac{t\,dt}{t} = -2t + c = -2\sqrt{a^2-x^2} + c$

Q.5 (C)

$\int x^{13/2}(1+x^{5/2})^{1/2}dx$

$1 + x^{5/2} = t^2 \Rightarrow x^{3/2}dx = \frac{4}{5}t\,dt$

$\int x^5.x^{3/2}(1 + x^{5/2})^{1/2}dx$

$= \frac{4}{5}\int (t^2-1)^2 . t^2\,dt = \frac{4}{5}\int (t^4 - 2t^2 + 1)t^2\,dt$

$= \frac{4}{5}\int (t^6 - 2t^4 + t^2)\,dt = \frac{4}{5}\left[\frac{t^7}{7} - \frac{2}{5}t^5 + \frac{t^3}{3}\right] + c$

$= \frac{4}{35}(1 + x^{5/2})^{7/2} - \frac{8}{25}(1 + x^{5/2})^{5/2} + \frac{4}{15}(1 + x^{5/2})^{3/2} + c$

Q.6 (C)

$\int\left(xe^{\ell n \sin x} - \cos x\right)dx$

$= \int x\sin x - \int \cos x\,dx$

$= -x\cos x + \int \cos x\,dx - \int \cos x\,dx$

$= -e^{\ell nx}\cos x + C$

Q.7 (A)

$\int\left\{\underbrace{\ell n(1+\sin x)}_{f(x)} + x\underbrace{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}_{f'(x)}\right\}dx$

$= x\,\ell n(1 + \sin x) + c$

Q.8 (A)

$I = \int \underbrace{\left(\frac{x}{\sqrt{1+x^2}}\right)}_{\text{II}} \underbrace{\ell n\left(x+\sqrt{1+x^2}\right)}_{\text{I}}dx$

$= \ell n(x + \sqrt{1+x^2})\int \frac{x}{\sqrt{1+x^2}}dx - \int \frac{1}{\left(x+\sqrt{1+x^2}\right)}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$

$I_1 = \int \frac{x}{\sqrt{1+x^2}}dx$

$1 + x^2 = t^2 \Rightarrow xdx = tdt$

$I = \ell n(x+\sqrt{1+x^2})\int \frac{t\,dt}{t} - \int\left(\frac{1}{\sqrt{1+x^2}}\int dt\right)dx$

$= \ell n(x + \sqrt{1+x^2}).\sqrt{1+x^2} - x + c$

Q.9 (A)

$\int \frac{dx}{x+x^5} = f(x) + c$

$= \int \frac{x^4+1-1}{x+x^5}\,dx = \int \frac{dx}{x} - \int \frac{dx}{x+x^5} = \ell n\, x - f(x) + c$

Q.10 (C)

$\int \frac{x^4+1}{x(x^2+1)^2}\,dx$

$\Rightarrow \int \left(\frac{x^4+2x^2+1-2x^2}{x(x^2+1)^2}\right)dx$

$\Rightarrow \int \frac{dx}{x} + \int \frac{dt}{t^2} \Rightarrow \ell n\, x + \frac{1}{1+x^2} + c$

Q.11 (A)

$\int \frac{1}{x\sqrt{1-x^3}}\,dx$

$1 - x^3 = t^2$ (Aliter put $x^3 = \sin^2\theta$)

$x^2\,dx = -\frac{2}{3}\,t\,dt$

$= \int \frac{x^2\,dx}{x^3\sqrt{1-x^3}}$

$= -\frac{2}{3}\int \frac{t\,dt}{(1-t^2)\,t} = \frac{2}{3}\int \frac{dt}{(t^2-1)}$

$= \frac{2}{3}\,\ell n\,\frac{t-1}{t+1} + c = \frac{1}{3}\,\ell n\,\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1} + c$

Q.12 (A)

$I = \int \frac{1}{[(x-1)^3(x+2)^5]^{1/4}}\,dx$

$= \int \frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}\,dx$ put $\frac{x-1}{x+2} = t$

$\Rightarrow \frac{3}{(x+2)^2}\,dx = dt$

$\Rightarrow I = \frac{1}{3}\int t^{-3/4}\,dt = \frac{1}{3}\,\frac{t^{1/4}}{(1/4)} + C$

$= \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$

Q.13 (A)

$\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx = \int \frac{1-\sqrt{x}}{\sqrt{1-x}}\,dx$

$= \int \frac{dx}{\sqrt{1-x}} - \int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$

Let $I_1 = \int \sqrt{\frac{x}{1-x}}\,dx = \int \frac{\sqrt{x}}{\sqrt{1-(\sqrt{x})^2}}\,dx$

put $\sqrt{x} = t \Rightarrow \frac{dx}{2\sqrt{x}} = dt$

$= \int \frac{2t^2\,dt}{\sqrt{1-t^2}} = 2\int \frac{t^2+1-1}{\sqrt{1-t^2}}\,dt$

$= 2\int \frac{dt}{\sqrt{1-t^2}} - 2\int \sqrt{1-t^2}\,dt$

$= 2\sin^{-1} t - 2\left[\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t\right] + c$

$I = -2\sqrt{1-x} - 2\sin^{-1}\sqrt{x} + \sqrt{x}\,\sqrt{1-x} + \sin^{-1}\sqrt{x} + c$

$= -2\sqrt{1-x} - \sin^{-1}\sqrt{x} + \sqrt{x}\,\sqrt{1-x} + c$

$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\,\sqrt{1-x} + c$

Q.14 (D)

$\int \sqrt{\frac{1-\cos x}{\cos\alpha - \cos x}}\,dx$

$= \int \sqrt{\frac{1-\left(1-2\sin^2\frac{x}{2}\right)}{\cos\alpha - \left(2\cos^2\frac{x}{2} - 1\right)}}\,dx$

$= \sqrt{2}\int \frac{\sin\frac{x}{2}}{\sqrt{\cos\alpha + 1 - 2\cos^2\frac{x}{2}}}\,dx$

$= \sqrt{2}\int \frac{\sin\frac{x}{2}}{\sqrt{2\cos^2\frac{\alpha}{2} - 2\cos^2\frac{x}{2}}}\,dx$

$= \int \frac{\sin\frac{x}{2}}{\sqrt{\cos^2\frac{\alpha}{2} - \cos^2\frac{x}{2}}}\,dx$

$\cos\frac{x}{2} = t \Rightarrow -\frac{1}{2}\sin\frac{x}{2}\,dx = dt$

$\sin\frac{x}{2}\,dx = -2dt$

$= -2\int\frac{dt}{\sqrt{\cos^2\frac{\alpha}{2}-t^2}} = -2\sin^{-1}\frac{t}{\cos\frac{\alpha}{2}} + c$

$= -2\sin^{-1}\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}} + c$

Q.15 (D)

$I = \int\frac{\sin x(1+\sin^2 x)}{2\cos^2 x - 1}$ put cos x = t = – sin x dx = dt

$I = -\int\frac{2-t^2}{2t^2-1}dt = \int\frac{t^2-2}{2t^2-1}dt = \frac{1}{2}\int\frac{2t^2-4}{2t^2-1}dt$

$= \frac{1}{2}\int dt - \frac{3}{2}\int\frac{dt}{2t^2-1}$

$= \frac{t}{2} - \frac{3}{2\sqrt{2}}\cdot\frac{1}{2}\ell n\left|\frac{\sqrt{2}t-1}{\sqrt{2}t+1}\right| + C$

$= \frac{1}{2}\cos x - \frac{3}{4\sqrt{2}}\ell n\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right| + C$

$A = \frac{1}{2}, B = \frac{-3}{4\sqrt{2}}, f(x) = \ell n\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right|$

or $A = \frac{1}{2}, B = \frac{3}{4\sqrt{2}}, f(x) = \ell n\left|\frac{\sqrt{2}\cos x+1}{\sqrt{2}\cos x-1}\right|$

Q.16 (A)

$2\int\frac{\sin x}{\sin 4x}dx$

$2\int\frac{\sin x}{2\sin 2x\cos 2x}dx = \int\frac{\sin x}{2\sin x\cos x\cos 2x}dx$

$= \frac{1}{2}\int\frac{dx}{\cos x\cos 2x} = \frac{1}{2}\int\frac{\cos x\,dx}{\cos 2x\cos^2 x}$

$= \frac{1}{2}\int\frac{\cos x\,dx}{(1-2\sin^2 x)(1-\sin^2 x)}$

put sin x = t $\Rightarrow$ cos x dx = dt

$= \frac{1}{2}\int\frac{dt}{(1-2t^2)(1-t^2)} = \frac{1}{2}\int\frac{dt}{(t^2-1)(2t^2-1)}$

$= \frac{1}{2}\int\frac{(2t^2-1)-2(t^2-1)}{(t^2-1)(2t^2-1)}dt$

$= \frac{1}{2}\int\frac{dt}{t^2-1} - \frac{1}{2}\int\frac{dt}{t^2-1/2}$

$= \frac{1}{4}\ell n\frac{t-1}{t+1} - \frac{1}{2.2\left(1/\sqrt{2}\right)}\ell n\frac{t-1/\sqrt{2}}{t+1/\sqrt{2}} + c$

$= \frac{1}{4}\ell n\frac{\sin x-1}{\sin x+1} - \frac{1}{2\sqrt{2}}\ell n\frac{\sqrt{2}\sin x-1}{\sqrt{2}\sin x+1} + c$

Q.17 (B)

$\int\frac{1+x^4}{(1-x^4)^{3/2}}dx$

$= \int\frac{1+x^4}{x^3\left(\frac{1}{x^2}-x^2\right)^{3/2}}dx = \int\frac{\frac{1}{x^3}+x}{\left(\frac{1}{x^2}-x^2\right)^{3/2}}dx$

Let $\frac{1}{x^2} - x^2 = t^2$

$\Rightarrow \left(\frac{1}{x^3}+x\right)dx = -t\,dt$

$= -\int\frac{t\,dt}{t^3} = -\left(\frac{t^{-2+1}}{-2+1}\right) + c$

$= \frac{1}{t} + c = \frac{1}{\sqrt{\frac{1}{x^2}-x^2}} + c$

Q.18 (B)

$f(x) = \int\frac{2\sin x - \sin 2x}{x^3}dx$

$f'(x) = \frac{2\sin x - \sin 2x}{x^3} = \frac{2\sin x}{x}\cdot\frac{1-\cos x}{x^2}$

$= 2\left(\frac{\sin x}{x}\right)\cdot\frac{2\sin^2\frac{x}{2}}{\frac{x^2}{4}\times 4} = \frac{2\times 2}{4}\left(\frac{\sin x}{x}\right)\cdot\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$

$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\cdot\left(\frac{\sin x}{x}\right)\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = 1$

JEE-ADVANCED
MCQ/COMPREHENSION/ COLUMN MATCHING

Q.1 (B, C)

$I = \int 2^{mx}.\,3^{nx}\,dx = \int (2^m.\,3^n)^x\,dx$

$= \frac{2^{mx}.\,3^{nx}}{\ell n(2^m.\,3^n)} + C$

Q.2 (B, C, D)

$\int \frac{x^2+\cos^2 x}{1+x^2}\,\text{cosec}^2 x\,dx$

$= \int \text{cosec}^2 x\,dx - \int \frac{1}{1+x^2}\,dx = -\cot x - \tan^{-1}x + C$

$= -\cot x - \cot^{-1}x + C$ .... (1)

$= -\tan^{-1}x - \frac{\text{cosec}\,x}{\sec x} + C$ .... (2)

$= -e^{\ell n \tan^{-1}x} - \cot x + C$ .... (3)

Q.3 (B,D)

$I = \int \frac{\ell n\left(\frac{x-1}{x+1}\right)}{x^2-1}\,dx$ put $\ell n\left(\frac{x-1}{x+1}\right) = t$

$\Rightarrow \frac{2}{x^2-1}\,dx = dt$

$\Rightarrow I = \int t\frac{dt}{2} = \frac{t^2}{4} + C = \frac{1}{4}\log^2\left(\frac{x-1}{x+1}\right) + C = \frac{1}{4}$

$\log^2\left(\frac{x+1}{x-1}\right) + C$

Q.4 (A, C, D)

$I = \int \frac{\ell n(\tan x)}{\sin x\,\cos x}\,dx$ put $\ell n \tan x = t$

$\Rightarrow \frac{1}{\sin x\cos x}\,dx = dt$

$\therefore I = \int t\,dt = \frac{t^2}{2} + C = \frac{1}{2}(\ell n \tan x)^2 + C$

$= \frac{1}{2}(\ell n \cot x)^2 + C = \frac{1}{2}(\ell n^2 \cot x) + C$

$= \frac{1}{2}\ell n^2(\sin x \sec x) + C = \frac{1}{2}\ell n^2(\cos x\,\text{cosec}\,x) + C$

Q.5 (A,C)

$\int \frac{\ell n(x+1) - \ell n\,x}{x(x+1)}\,dx$

$\int \frac{\ell n\left(\frac{x+1}{x}\right)}{x(x+1)}\,dx$

Let $\ell n\left(\frac{x+1}{x}\right) = t \quad \frac{1}{\left(\frac{x+1}{x}\right)}\frac{x-(x+1)}{x^2}\,dx = dt$

$\frac{dx}{x(x+1)} = -dt$

$= -\int t\,dt = -\frac{t^2}{2} + c \quad = -\frac{1}{2}\left[\ell n^2\left(\frac{x+1}{x}\right)\right] + c$

$= -\frac{1}{2}[\ell n(x+1) - \ell n\,x]^2 + c$

$= -\frac{1}{2}[\ell n^2(x+1) + \ell n^2(x) - 2\,\ell n\,x\,.\,\ell n(x+1)] + c$

$= -\frac{1}{2}\ell n^2(x+1) - \frac{1}{2}\ell n^2(x) + \ell n\,x\,\ell n(x+1) + c$

$= -\frac{1}{2}\ell n^2\left(1+\frac{1}{x}\right) + c$

Q.6 (C,D)

$\int e^{3x}\cos 4x\,dx = e^{3x}(A\sin 4x + B\cos 4x) + c$

Diff. both the side w.r.t. x

$e^{3x}\cos 4x = 3e^{3x}(A\sin 4x + B\cos 4x)$
$+ e^{3x}(4A\cos 4x - 4B\sin 4x)$

$\cos 4x = (3A\sin 4x + 3B\cos 4x) + 4A\cos 4x - 4B\sin 4x$

$1 = 3B + 4A$

$3A - 4B = 0$

$3A = 4B$

Q.7 (A,B,D)

$I = \int \frac{dx}{\sqrt{x-x^2}} = \int \frac{dx}{\sqrt{\frac{1}{4} - \left(x-\frac{1}{2}\right)^2}}$

$= \sin^{-1}\frac{\left(x-\frac{1}{2}\right)}{1/2} = \sin^{-1}(2x-1) + C$ ..... (1)

Also $I = \int \frac{dx}{\sqrt{x}\,\sqrt{1-x}}$ put $\sqrt{x} = t$

$\Rightarrow \frac{1}{2\sqrt{x}}\ dx = dt$

$\Rightarrow I = \int \frac{2\,dt}{\sqrt{1-t^2}} = 2\sin^{-1}\sqrt{x} + C$ ....(2)

Now Let $\theta = \sin^{-1}(2x-1)$

$\Rightarrow \sin\theta = 2x - 1$

$\Rightarrow \cos\theta = \sqrt{1-(2x-1)^2}$

$= \sqrt{1-4x^2+4x-1} = 2\sqrt{x-x^2}$

$\Rightarrow \theta = \cos^{-1} 2\sqrt{x-x^2}$

Q.8 (A,C)

$\int \frac{x-1}{x^2\sqrt{2x^2-2x+1}}\,dx = \frac{\sqrt{f(x)}}{g(x)} + c$

put $x = 1/t \Rightarrow dx = -\frac{dt}{t^2}$

$= \int \frac{\frac{1}{t}-1}{\frac{1}{t^2}\sqrt{\frac{2}{t^2}-\frac{2}{t}+1}}\left(-\frac{dt}{t^2}\right)$

$= \int \frac{(t-1)\,dt}{\sqrt{t^2-2t+2}} = \frac{1}{2}\int \frac{(2t-2)\,dt}{\sqrt{t^2-2t+2}}$

Let $t^2 - 2t + 2 = z^2 \Rightarrow (2t-2)\,dt = 2z\,dz$

$= \frac{1}{2}\int \frac{2z\,dz}{z}$

$= z + c$

$= \sqrt{t^2-2t+2} + c$

$= \sqrt{\frac{1}{x^2}-\frac{2}{x}+2} + c$

$= \frac{\sqrt{2x^2-2x+1}}{x} + c$

$g(x) = x$

$f(x) = 2x^2 - 2x + 1$

Q.9 (A, B)

Let $I = \int \frac{dx}{5+4\cos x} = I\tan^{-1}\left(\tan\frac{x}{2}\right) + C$

$\int \frac{dx}{1+4(1+\cos x)} = \int \frac{\sec^2\frac{x}{2}dx}{5+5\tan^2\frac{x}{2}+4-4\tan^2\frac{x}{2}}$

$= \int \frac{\sec^2\frac{x}{2}dx}{9+\tan^2\frac{x}{2}}$ Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}\ dx = 2dt$

$\Rightarrow I = 2\int \frac{dt}{9+t^2} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + C \Rightarrow I = \frac{2}{3}$, m

$= \frac{1}{3}$

Q.10 (A, B, C, D)

$I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x}\ dx = \int \frac{2\tan x\sec^2 x}{\tan^4 x+1}dx$ put

$\tan^2 x = t \Rightarrow 2\tan x\sec^2 x\,dx = dt$

$= \int \frac{dt}{t^2+1} = \tan^{-1}(t) + C$

$= \tan^{-1}(\tan^2 x) + C \qquad = \frac{\pi}{2} - \cot^{-1}(\tan^2 x) + C$

$= -\cot^{-1}(\tan^2 x) + C_1 = -\cot^{-1}\left(\frac{1}{\cot^2 x}\right) + C_1$

$= \cot^{-1}(\cot^2 x) + C_1$

also $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x} \Rightarrow \left(\frac{1-\cos 2x}{1+\cos 2x}\right) = \tan^2 x$,

using these values in given integral

$I = \int \frac{\sin 2x}{(\cos^2 x - \sin^2 x)^2 + 2\sin^2 x\cos^2 x}dx$

$= \int \frac{\sin 2x}{(\cos 2x)^2 + \left(\frac{1-\cos^2 2x}{2}\right)}dx$

$= \int \frac{2\sin 2x}{(\cos 2x)^2+1}dx$ put $\cos 2x = t$

$\Rightarrow -2\sin 2x\,dx = dt$

$\Rightarrow I = \int \frac{-dt}{t^2+1} = -\tan^{-1} t + C_2 = -\tan^{-1}(\cos 2x) + C_2$

Q.11 (A, C, D)

$f'(x) = 3x^2\sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right)$

Now $\lim_{x\to 0} f'(x) = 0 - 0 = 0$

$\therefore$ at $x = 0$, $f(x)$ is derivable so continuous also, and $f'(x)$ is continuous at $x = 0$

now $f''(x) = 6x \sin\left(\frac{1}{x}\right) + 3x^2 \cos\left(\frac{1}{x}\right).\frac{1}{x^2} - \frac{1}{x}\sin \frac{1}{x} - \cos\left(\frac{1}{x}\right)$

so $f''(x) = 6x \sin\left(\frac{1}{x}\right) - \frac{1}{x}\sin\left(\frac{1}{x}\right) + 2\cos\left(\frac{1}{x}\right)$

at x = 0, $\cos\left(\frac{1}{x}\right)$ is not define so at x = 0, $f''(x)$ is not continuous or $f'(x)$ in not differentiable x = 0.

Q.12 (A, B)

$I_n = \int \cot^n x\, dx = \int \cot^{n-2} x(\text{cosec}^2 x - 1)\, dx$

$= \frac{-\cot^{n-1}x}{n-1} - I_{n-2} + C_1$

$\Rightarrow I_n + I_{n-2} = \frac{-\cot^{n-1}x}{n-1} + C_1$

Now, $I_0 + I_1 + 2(I_2 + ...+ I_8) + I_9 + I_{10}$
$= (I_2 + I_0) + (I_3 + I_1) + (I_4 + I_2) + (I_5 + I_3) + (I_6 + I_4) + (I_7 + I_5) + (I_8 + I_6) + (I_9 + I_7) + (I_{10} + I_8)$

$= -\left(\frac{\cot x}{1} + \frac{\cot^2 x}{2} + ... + \frac{\cot^9 x}{9}\right) + C$

$\therefore$ A = – 1.

Paragraph for question nos. 13 & 14

Q.13 (A)

Q.14 (B)

(13,14) $f(x) = ax^2 + bx + c$
and $g(x) = px^2 + qx + r$
$g(0) = 2 \Rightarrow r = 2$
$g'(x) = 2px + q$
$\Rightarrow g'(0) = -3 \Rightarrow q = -3$
$g''(x) = 2p \Rightarrow p = 1$
$\therefore g(x) = x^2 - 3x + 2$
Again, $a^2 + b^2 + c^2 - 2a + 4b - 2c + 6 = 0$
$(a - 1)^2 + (b + 2)^2 + (c - 1)^2 - 6 + 6 = 0$
Hence, a = 1; b = – 2; c = 1
$\therefore f(x) = x^2 - 2x + 1 = (x - 1)^2$

(i) $f(1) + g(1) = 0 + 0 = 0$

(ii) $\int \frac{f(x)}{g(x)} dx = \int \frac{(x-1)^2}{x^2 - 3x + 2} dx$

$= \int \frac{(x-1)^2}{(x-1)(x-2)} dx = \int \frac{x-1}{x-2} dx$

$= \int \frac{x-2+1}{x-2} dx = \int \left(1 + \frac{1}{x+2}\right) dx$

$= x + \ln|x - 2| + C$. **Ans.**]

Paragraph for question nos. 15 to 17

Q.15 (D)

Q.16 (C)

Q.17 (D)

(i) $\int \frac{f(x)}{x^3 - 1} dx = \ln\left|\frac{x^2 + x + 1}{x - 1}\right| + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

Differentiating both sides, we get

$\frac{f(x)}{x^3 - 1}$

$= \frac{x-1}{x^2+x+1} \frac{(x-1)(2x+1) - (x^2+x+1).1}{(x-1)^2} + \frac{2}{\sqrt{3}} . \frac{1}{1 + \left(\frac{2x+1}{\sqrt{3}}\right)^2} . \frac{2}{\sqrt{3}}$

$= \frac{2x^2 - x - 1 - x^2 - x - 1}{x^3 - 1} + \frac{4}{3} . \frac{3}{3 + (4x^2 + 4x + 1)}$

$= \frac{x^2 - 2x - 2}{x^3 - 1} + \frac{1}{x^2 + x + 1}$

$f(x) = (x^2 - 2x - 2) + (x - 1) = x^2 - x - 3 \Rightarrow f(1) = -3$ **Ans.**

(ii) $I = \int \frac{1 - 6\text{cosec } x}{6 + f(\sin x)} d(\sin x)$

$= \int \frac{1 - \frac{6}{\sin x}}{6 + \sin^2 x - \sin x - 3} d(\sin x)$

$$\int \frac{1-\dfrac{6}{\sin x}}{\sin^2 x-\sin x+3}\, d(\sin x) \quad \text{(Put } \sin x = t)$$

[T/S, inde] done

$$= \int \frac{1-\dfrac{6}{t}}{t^2-t+3}dt = \int \frac{\dfrac{1}{t^2}-\dfrac{6}{t^3}}{1-\dfrac{1}{t}+\dfrac{3}{t^2}}dt$$

$$= \ln\left(1-\frac{1}{t}+\frac{3}{t^2}\right)+K$$

$$= \ln\left(1-\frac{1}{\sin x}+\frac{3}{\sin^2 x}\right)+K$$

$$\Rightarrow \quad g(x) = \ln\left(1-\frac{1}{\sin x}+\frac{3}{\sin^2 x}\right)$$

$$\therefore \quad g(t) = \ln\left(1-\frac{1}{\sin t}+\frac{3}{\sin^2 t}\right)$$

Now, $\lim\limits_{t\to\frac{\pi}{2}} g(t) = \ln 3$ **Ans.**

(iii) $I = \int \frac{5+f(\sin x)+f(\cos x)}{\sin x+\cos x}dx$

$$= \int \frac{5+\sin^2 x-\sin x-3+\cos^2 x-\cos x-3}{\sin x+\cos x}dx$$

$$= \int -dx = -x+\lambda$$

$\therefore\ h(x) = -x \quad (\text{since } h(1) = -1)$

Hence $\tan^{-1}(h(2)) + \tan^{-1}(h(3)) = \tan^{-1}(-2)$

$+ \tan^{-1}(-3) = \dfrac{-3\pi}{4}$ **Ans.**]

Q.18 (A) → (p), (B) → (p), (C) → (r), (D) → (s)

(A) $F(x) = \int \frac{x+\sin x}{1+\cos x}dx = \int\left(x\,\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)dx$

$= x\tan x/2 + C$

Since $0 = F(0)$

$\therefore$ C = 0 and $F(\pi/2) = \pi/2$

(B) $F(x) = \int e^{\sin^{-1}x}\left(1-\frac{x}{\sqrt{1-x^2}}\right)dx$

$$= \int e^{\sin^{-1}x}\left(\frac{1}{\sqrt{1-x^2}}\sqrt{1-x^2}+\frac{-x}{\sqrt{1-x^2}}\right)dx$$

put $\sin^{-1}x = t \Rightarrow \frac{dx}{\sqrt{1-x^2}}$ dt

$\Rightarrow F(x) = \int e^t\left(\sqrt{1-\sin^2 t}-\sin t\right)dt = e^t\cos t + C$

$= e^{\sin^{-1}x}\sqrt{1-x^2} + C$

$\because 1 = F(0) \Rightarrow C = 0$

Hence, $F(1/2) = e^{\pi/6}.\frac{\sqrt{3}}{2} = \frac{k\sqrt{3}}{\pi}e^{\pi/6}$ (given)

$\therefore k = \frac{\pi}{2}$

(C) $F(x) = \int \frac{dx}{(x^2+1)(x^2+9)} = \frac{1}{8}\int\left(\frac{1}{x^2+1}-\frac{1}{x^2+9}\right)dx$

$= \frac{1}{8}\left[\tan^{-1}x-\frac{1}{3}\tan^{-1}\frac{x}{3}\right]+C$

$0 = F(0) = C$

$\therefore \frac{1}{8}\left(\frac{\pi}{3}-\frac{1}{3}.\frac{\pi}{6}\right) = \frac{5\pi}{144} = \frac{5k}{36}$

$\therefore k = \frac{\pi}{4}$

(D) $F(x) = \int \frac{\sqrt{\tan x}}{\sin x\cos x}dx = \int(\tan x)^{\frac{-1}{2}}\sec^2 x\,dx = 2\sqrt{\tan x}$ $+ C$

$\because 0 = F(0) \Rightarrow C = 0$

$\therefore F(\pi/4) = 2 = \frac{2k}{\pi}$

$\therefore k = \pi$

Q.19 (A) → (s) ; (B) → (q) ; (C) → (r)

(A) $v = 0 \Rightarrow a = b$, Also $a + b = u \Rightarrow a = \frac{u}{2}$

Now, $I = \frac{1}{a}\int\frac{dx}{1+\cos x}$

$$= \frac{2}{u}\int\frac{dx}{2\cos^2\frac{x}{2}} = \frac{2}{u}\int\frac{1}{2}\sec^2\frac{x}{2}dx$$

$$= \frac{2}{u}\tan\frac{x}{2} + C$$

(B) $v > 0 \Rightarrow a > b$

Now, $I = \int\frac{dx}{a+b\cos x} = \int\frac{dx}{(a-b)+2b\cos^2\frac{x}{2}}$

$$= \int\frac{\sec^2\frac{x}{2}dx}{(a+b)+(a-b)\tan^2\frac{x}{2}}$$

put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$\Rightarrow I = \int \frac{2dt}{(a+b)+(a-b)+t^2}$$

$$= \frac{2}{a-b}\int \frac{dt}{t^2+\left(\frac{a+b}{a-b}\right)} \quad ...(1)$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\,t\right)+C$$

$$= \frac{2}{\sqrt{uv}}\tan^{-1}\left(\sqrt{\frac{v}{u}}\tan\frac{x}{2}\right)+C$$

(C) $v < 0 \Rightarrow a-b < 0 \Rightarrow b-a > 0$

Now $I = \frac{2}{b-a}\int \frac{dt}{\frac{a+b}{b-a}-t^2}$ (using equation (1) of part (B))

$$= \frac{2}{b-a}\,\frac{1}{2}\sqrt{\frac{b-a}{b+a}}\ln\left|\frac{\sqrt{\frac{b+a}{b-a}}+t}{\sqrt{\frac{b+a}{b-a}}-t}\right|+C$$

$$= \frac{1}{\sqrt{-uv}}\ln\left|\frac{\sqrt{u}+\sqrt{-v}\tan\frac{x}{2}}{\sqrt{u}-\sqrt{-v}\tan\frac{x}{2}}\right|+C$$

NUMERICAL VALUE BASED

Q.1 [1]

$$f(x) = \int \frac{2\sin x-\sin 2x}{x^3}\,dx$$

$$f'(x) = \frac{2\sin x-\sin 2x}{x^3} = \frac{2\sin x}{x}\cdot\frac{1-\cos x}{x^2}$$

$$= 2\left(\frac{\sin x}{x}\right)\cdot\frac{2\sin^2\frac{x}{2}}{\frac{x^2}{4}\times 4} = \frac{2\times 2}{4}\left(\frac{\sin x}{x}\right)\cdot\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\cdot\left(\frac{\sin x}{x}\right)\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = 1$$

Q.2 [3]

$$\int \sin^4 x\cos^4 x\,dx = \frac{1}{16}\int \sin^4 2x\,dx$$

$$= \frac{1}{16}\int\left(\frac{1-\cos 4x}{2}\right)^2 dx$$

$$= \frac{1}{64}\int\left(1+\cos^2 4x-2\cos 4x\right)dx$$

$$= \frac{1}{64}\int dx - \frac{1}{32}\int \cos 4x\,dx + \frac{1}{64}\int\left(\frac{1+\cos 8x}{2}\right)dx$$

$$= \frac{x}{64}-\frac{\sin 4x}{128}+\frac{1}{64}\times\frac{x}{2}+\frac{1}{128}\,\frac{\sin 8x}{8}+C$$

$$= \frac{3}{128}x-\frac{1}{128}\sin 4x+\frac{\sin 8x}{128\times 8}+C$$

$$= \frac{1}{128}\left[3x-\sin 4x+\frac{\sin 8x}{8}\right]+C$$

Q.3 [12]

$f(x) = \int \frac{3x+2}{\sqrt{x-9}}\,dx$ put $x-9 = t^2$

$$= 2\int (29+3t^2)\,dt$$

$f(x) = 2(29\sqrt{x-9} + (x-9)^{3/2}) + C$

$\because$ $f(10) = 60$ $\Rightarrow$ $C = 0$

$f(x) = 2(29\sqrt{x-9} + (x-9)^{3/2})$

$f(13) = 132$ $\Rightarrow$ sum of digits = 6

2(sum of digits = 12

Q.4 [10]

$$I = \int \frac{\sqrt{4+x^2}}{x^6}\,dx$$

Put $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\,d\theta$

$$\Rightarrow I = \int \frac{2\sec\theta.2\sec^2\theta\,d\theta}{2^6\tan^6\theta}$$

$$= \frac{1}{2^4}\int \frac{\cos^3\theta\,d\theta}{\sin^6\theta}$$

$$= \frac{1}{2^4}\int\left(\frac{1-\sin^2\theta}{\sin^6\theta}\right)\cos\theta\,d\theta$$

put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

$$= \frac{1}{2^4}\int \frac{dt}{t^6} - \int \frac{1}{t^4}dt = \frac{1}{16}\left[\frac{1}{3t^3} - \frac{1}{5t^5}\right] + C$$

$$= \frac{1}{16}\left(\frac{5\sin^2\theta - 3}{15\sin^5\theta}\right) + C$$

but $\tan\theta = \frac{x}{2}$ So $\sin\theta = \frac{x}{\sqrt{4+x^2}}$

$$= \frac{1}{16}\left(\frac{\frac{5x^2}{x^2+4} - 3}{\frac{15x^5}{(x^2+4)^{5/2}}}\right) + C$$

$$= \frac{1}{16}\left(\frac{(2x^2-12)(x^2+4)^{3/2}}{15x^5}\right) + C$$

$$= \frac{1}{120}\left(\frac{(x^2+4)^{3/2}(x^2-6)}{x^5}\right) + C$$

Q.5 [5]

$I = \int \sqrt{\frac{x}{a^3 - x^3}}\, dx$ put $x^3 = a^3\sin^2\theta$

$\Rightarrow x = a\sin^{2/3}\theta \Rightarrow dx = a \times \frac{2}{3}\frac{\cos\theta}{\sin^{1/3}\theta}\, d\theta$

$$\Rightarrow I = \int \sqrt{\frac{a\sin^{2/3}\theta}{a^3(1-\sin^2\theta)}} \times \frac{2a}{3}\left(\frac{\cos\theta}{\sin^{1/3}\theta}\right) d\theta$$

$$= \int \frac{\sin^{1/3}\theta}{a\cos\theta} \times \frac{2a}{3}\frac{\cos\theta}{\sin^{1/3}\theta}\, d\theta = \frac{2}{3}\theta + C$$

$$= \frac{2}{3}\sin^{-1}\left(\left(\frac{x}{a}\right)^{3/2}\right) + C$$

Q.6 [2]

$$I = \int \frac{x\, dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$$

Put $1 + x^2 = t^2$, then $2x\, dx = 2t\, dt$

$$\Rightarrow I = \int \frac{t\, dt}{\sqrt{t^2+t^3}} = \int \frac{dt}{\sqrt{1+t}}$$

$$= 2\sqrt{1+t} = 2\sqrt{1+\sqrt{1+x^2}} + C$$

Q.7 [2]

$$\int e^{\sin x}.\frac{x\cos^3 x - \sin x}{\cos^2 x}\, dx$$

$$= \int e^{\sin x}(x\cos x - \tan x\sec x)\, dx$$

$$= \int e^{\sin x}.x\cos x\, dx - \int e^{\sin x}\tan x\sec x\, dx$$

(by parts integration)

$$= xe^{\sin x} - \int 1.e^{\sin x}dx - [e^{\sin x}\sec x - \int e^{\sin x}\cos x.\sec x\, dx] + C$$

$= e^{\sin x}(x - \sec x) + C \quad \therefore \quad f(x) = x - \sec x$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2$$

Q.8 [16]

$g(x) = \int \frac{1+2\cos x}{(\cos x+2)^2}dx$;

$$g(x) = \int \frac{\cos x(\cos x+2)+\sin^2 x}{(\cos x+2)^2}dx$$

$$g(x) = \int \cos x.\frac{1}{(\cos x+2)}dx + \int \frac{\sin^2 x}{(\cos x+2)^2}dx$$

$$= \frac{1}{(\cos x+2)}.\sin x - \int -\frac{(-\sin x)}{(\cos x+2)^2}.\sin x\, dx + \int \frac{\sin^2 x}{(\cos x+2)^2}dx$$

$$g(x) = \frac{\sin x}{(\cos x+2)} + C$$

$g(0) = 0$

$\Rightarrow \quad C = 0 \quad g(x) = \frac{\sin x}{\cos x+2}$

$g\left(\frac{\pi}{2}\right) = \frac{1}{2}$; $32g\left(\frac{\pi}{2}\right) = 16$

Q.9 [12]

$fog(x) = \sqrt{e^x - 1}$

$$\int fog(x)dx = \int \sqrt{e^x - 1}\, dx$$

put $e^x - 1 = t^2$ $= \int \frac{2t^2}{1+t^2}dt = 2t - 2\tan^{-1}t + C$

$= 2\sqrt{e^x - 1} - 2\tan^{-1}\sqrt{e^x - 1} + C$

$A^3 + B^2 = 2^3 + (-2)^2 = 12$

Q.10 [11]

$I = \int \dfrac{2\sin 2\phi - \cos\phi}{6 - \cos^2\phi - 4\sin\phi}\, d\phi$

$= \int \dfrac{(4\sin\phi - 1)\cos\phi}{6 - (1 - \sin^2\phi) - 4\sin\phi}\, d\phi$

put $\sin\phi = t \Rightarrow \cos\phi\, d\phi = dt$

$\Rightarrow I = \int \dfrac{(4t - 1)\,dt}{5 + t^2 - 4t} = 2\int \dfrac{(2t - 4) + 7/2}{t^2 - 4t + 5}dt$

$= 2\,[\ell n|t^2 - 4t + 5|] + 7\int \dfrac{1}{(t-2)^2 + (1)^2}dt$

$= 2\,[\ell n|t^2 - 4t + 5|] + 7\tan^{-1}\left(\dfrac{t-2}{1}\right) + C = 2$

$\ell n\left|\sin^2\phi - 4\sin\phi + 5\right| + 7\tan^{-1}(\sin\phi - 2) + C$

Q.11 [13]

$I = \int \dfrac{(x-1)^2}{x^4 + x^2 + 1}dx$

$= \int \dfrac{x^2 + 1}{x^4 + x^2 + 1}dx - \int \dfrac{2x}{x^4 + x^2 + 1}dx = I_1 - I_2$

Now $I_1 = \int \dfrac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3}dx$

put $x - \dfrac{1}{x} = t \Rightarrow \left(1 + \dfrac{1}{x^2}\right)dx = dt$

$\Rightarrow I_1 = \int \dfrac{dt}{t^2 + 3} = \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x - \frac{1}{x}}{\sqrt{3}}\right) + C_1$

Also $I_2 = \int \dfrac{2x}{x^4 + x^2 + 1}dx$ put $x^2 = t \Rightarrow 2x\,dx = dt$

$\Rightarrow I_2 = \int \dfrac{dt}{t^2 + t + 1} = \int \dfrac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x^2 + 1}{\sqrt{3}}\right) + C_2$

$\int \dfrac{(x-1)^2}{x^4 + x^2 + 1}dx = \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x^2 - 1}{x\sqrt{3}}\right) - \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x^2 + 1}{\sqrt{3}}\right) + c$

Q.12 [1]

$\int \dfrac{1 + x\cos x}{x\left(1 - x^2 e^{2\sin x}\right)}dx$

Put $xe^{\sin x} = t \Rightarrow (xe^{\sin x}.\cos x + e^{\sin x})\,dx = dt$

$\Rightarrow e^{\sin x}(x\cos x + 1)\,dx = dt$

$\Rightarrow I = \int \dfrac{dt}{t(1 - t^2)} = \int \dfrac{1}{t(1-t)(1+t)}dt$

Let $\dfrac{1}{t(1-t)(1+t)} = \dfrac{A}{t} + \dfrac{B}{(1-t)} + \dfrac{C}{1+t}$

$1 = A(1-t)(1+t) + B(t)(1+t) + C(t)(1-t)$

put $t = 0 \Rightarrow$ $A = 1$

Put $t = 1 \Rightarrow$ $B = 1/2$

Put $t = -1 \Rightarrow$ $C = -1/2$

$\Rightarrow I = \int \left\{\dfrac{1}{t} + \dfrac{1}{2(1-t)} - \dfrac{1}{2(1+t)}\right\}dt$

$= \ell n|t| - \dfrac{1}{2}\ell n|1-t| - \dfrac{1}{2}\ell n|1+t| + C$

$= \ell n|x\, e^{\sin x}| - \dfrac{1}{2}\log|1 - x^2 e^{2\sin x}| + C = \ell n\sqrt{\dfrac{x^2 e^{2\sin x}}{1 - x^2 e^{2\sin x}}} + C$

Q.13 [2]

$\int \dfrac{x^4 + 1}{x(x^2 + 1)^2}dx = A\,\ell n|x| + \dfrac{B}{1 + x^2} + C$

$\int \dfrac{x^4 + 1}{x(x^2 + 1)^2}dx = \int \dfrac{(x^2 + 1)^2 - 2x^2}{x(x^2 + 1)^2}dx$

$= \int \dfrac{1}{x}dx - \int \dfrac{2x}{(x^2 + 1)^2}dx = \ell n|x| + \dfrac{1}{1 + x^2} + C$

Q.14 [1]

$I = \int \dfrac{1}{1 - \sin^4 x}dx = \int \dfrac{\sec^4 x}{\sec^4 x - \tan^4 x}dx$

$= \int \frac{\sec^2 x.\sec^2 x\,dx}{\sec^2 x + \tan^2 x}$

$= \int \frac{(1+\tan^2 x)\sec^2 x\,dx}{2\tan^2 x + 1}$

put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$\Rightarrow \quad I = \frac{1}{2}\int \frac{2+2t^2}{2t^2+1}dt$

$= \frac{1}{2}\left[\int dt + \int \frac{1}{2t^2+1}dt\right]$

$= \frac{1}{2}t + \frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\,t) + C$

$= \frac{1}{2}\tan x + \frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + C$

Q.15 [11]

$I = \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}dx$

$= \int \frac{(2-3\sin^2 x + \sin^4 x)\cos x}{\sin^2 x(1+\sin^2 x)}dx$

Let sinx = t

$I = \int \frac{2-3t^2+t^4}{t^2+t^4} = \int\left(1+\frac{2}{t^2}-\frac{6}{t^2+1}\right)dt$

$= t - \frac{2}{t} - 6\tan^{-1}(t) + c$

$= \sin x - \frac{2}{\sin x} - 6\tan^{-1}(\sin x) + C$

Q.16 [10]

$I = \int \frac{dx}{\sqrt{\sin^3 x\cos^5 x}}$

$= \int \frac{dx}{\sqrt{\tan^3 x\cos^8 x}} = \int \frac{\sec^4 x\,dx}{\sqrt{\tan^3 x}}$

$= \int \frac{(1+\tan^2 x)\sec^2 x}{\sqrt{\tan^3 x}}dx \qquad$ Let $\tan x = t^{2/3}$

$\Rightarrow \sec^2 x\,dx = \frac{2}{3}t^{-1/3}dt$

$\Rightarrow \quad I = \int \frac{(1+t^{4/3})}{t}\frac{2}{3}t^{-1/3}dt$

$= \frac{2}{3}\int (1+t^{4/3}).t^{-4/3}dt$

$= \frac{2}{3}\int (t^{-4/3}+1)dt = \frac{2}{3}\left[\frac{t^{-1/3}}{(-1/3)}+t\right]+C$

$= -2t^{-1/3} + \frac{2}{3}t + C = -2\sqrt{\cot x} + \frac{2}{3}\tan^{3/2}x + C$

$\Rightarrow \quad a = -2, \; b = 2/3$

JEE-MAIN

PREVIOUS YEAR'S

Q.1 (2)

$\int \frac{2\sin^2\theta\cos 2\theta(\sin^6\theta+\sin^4\theta+\sin^2\theta)\sqrt{2\sin^4\theta+3\sin^2\theta+6}}{2\cos^2\theta}d\theta$

Let $\sin\theta = t$ $\cos\theta\,d\theta = dt$

$= \int (t^6+t^4+t^2)\sqrt{2t^4+3t^2+6}\;dt$

$= \int (t^5+t^3+t)\sqrt{2t^6+3t^4+6t^2}\;dt$

Let $2t^6+3t^4+6t^2 = z$

$12(t^5+t^3+t)\,dt = dz$

$= \frac{1}{12}\int \sqrt{z}\,dz = \frac{1}{18}z^{3/2} + c$

$= \frac{1}{18}(2\sin^6\theta + 3\sin^4\theta + 6\sin^2\theta)^{3/2} + C$

$= \frac{1}{18}[(1-\cos^2\theta)(2(1-\cos^2\theta)+3-3\cos^2\theta+6)]^{3/2} + C$

$= \frac{1}{18}[(1-\cos^2\theta)(2\cos^4\theta - 7\cos^2\theta + 11)]^{3/2} + C$

$= \frac{1}{18}[-2\cos^6\theta + 9\cos^4\theta - 18\cos^2\theta + 11]^{3/2} + C$

Q.2 (1)

put $\sin\theta + \cos\theta = t \Rightarrow \quad 1 + \sin 2\theta = t^2$

$\Rightarrow (\cos\theta - \sin\theta)\,d\theta = dt$

$\therefore I = \int \frac{dt}{\sqrt{8-(t^2-1)}} = \int \frac{dt}{\sqrt{9-t^2}} = \sin^{-1} + C$

$= \sin^{-1}\left(\frac{\sin\theta+\cos\theta}{3}\right) + C$

$\Rightarrow a = 1$ and $b = 3$

Q.3 [6]

$\int \frac{(x^2-1)dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)} + \int \frac{dx}{x^4+3x^2+1}$

$$\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left\{\left(x+\frac{1}{x}\right)^2+1\right\}\tan^{-1}\left(x+\frac{1}{x}\right)} + \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)dx}{x^4+3x^2+1}$$

Put $\tan^{-1}\left(\quad +\frac{1}{x}\right) =$

$$\int \frac{dt}{t} + \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+5} - \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2+1}$$

Put $x-\frac{1}{x} = y$, $x+\frac{1}{x} = z$

$$\log_e t + \frac{1}{2}\int \frac{dy}{y^2+5} - \frac{1}{2}\int \frac{dz}{z^2+1}$$

$$= \log_e \tan^{-1}\left(x+\frac{1}{x}\right) + \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right)$$

$$-\frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

$\alpha = 1$, $\beta = \frac{1}{2\sqrt{5}}$, $\gamma = \frac{1}{\sqrt{5}}$, $\delta = \frac{-1}{2}$

or

$\alpha = 1$, $\beta = \frac{-1}{2\sqrt{5}}$, $\gamma = \frac{-1}{\sqrt{5}}$, $\delta = \frac{-1}{2}$

$10(\alpha + \beta\gamma + \delta) = 10\left(1+\frac{1}{10}-\frac{1}{2}\right) = 6$

Q.4 (1)

$$\int \frac{(2x-1)\cos\sqrt{(2x-1)^2+5}}{\sqrt{(2x-1)^2+5}}\,dx$$

$(2x-1)^2 + 5 = t^2$

$2(2x-1)\,2dx = 2t\,dt$

$2\sqrt{t^2-5}\,dx = t\,dt$

So $\int \frac{\sqrt{t^2-5}\cos t}{2\sqrt{t^2-5}}\,dt = \frac{1}{2}\sin t + c$

$= \frac{1}{2}\sin\sqrt{\;2x-1^{2}}\;(+5+c)$

Q.5 [4]

$$(f)x = \int \frac{(5x^8+7x^6)\,dx}{x^{14}(x^{-5}+x^{-7}+2)^2}$$

Let $x^{-5} + x^{-7} + 2 = t$

$(-5x^{-6} - 7x^{-8})dx = dt$

$\Rightarrow f(x) =$

$\Rightarrow f(x) = \int -\frac{dt}{t^2} = \frac{1}{t} + c$

$f(x) = \frac{x^7}{x^2+1+2x^7}$

$f(1) = \frac{1}{4}$

Q.6 [3]

Q.7 (3)

Q.8 [80]

Q.9 [17]

JEE-ADVANCED PREVIOUS YEAR'S

Q.1 **(C)**

Put secx + tanx = t

$(\sec x\tan x + \sec^2 x)\,dx = dt$

$\sec x \,.\, t\,dx = dt$

$\sec x - \tan x = \frac{1}{t}$

$\sec x = \frac{t+\frac{1}{t}}{2}$

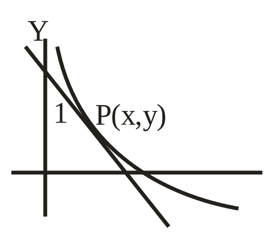
$$\int \frac{\sec x.dt}{t^{9/2}.t} = \int \frac{1}{2}\frac{\left(t+\frac{1}{t}\right)}{t.t^{9/2}}dt$$

$$= \frac{1}{2}\int\left(\frac{1}{t^{9/2}}+\frac{1}{t^{13/2}}\right)dt$$

$$= -\frac{1}{2}\left[\frac{2}{7t^{7/2}}+\frac{2}{11t^{11/2}}\right]+k$$

$$= -\frac{1}{t^{11/2}}\left[\frac{t^2}{7}+\frac{1}{11}\right]+k$$

Q.2 (A,D)



$Y - y = y'(X - x)$

So, $Y_p = (0, y - xy')$

So, $x^2 + (xy')^2 = 1 \Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-x^2}{x^2}}$

[$\frac{dy}{dx}$ can not be positive i.e. $f(x)$ can not be increasing in first quadrant, for $x \in (0,1)$]

Hence, $\int dy = -\int \frac{\sqrt{1-x^2}}{x} dx$

$\Rightarrow y = -\int \frac{\cos^2\theta \, d\theta}{\sin\theta}$; put $x = \sin\theta$

$\Rightarrow y = -\int \operatorname{cosec}\theta \, d\theta + \int \sin\theta \, d\theta$

$\Rightarrow y = \ln(\operatorname{cosec}\theta + \cot\theta) - \cos\theta + C$

$\Rightarrow y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2} + C$

$\Rightarrow y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$ (as $y(1) = 0$)

Definite Integration

EXERCISES

ELEMENTRY

Q.1 (1)

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

Also as $x = 0, \theta = 0$ and $x = 1, \theta = \frac{\pi}{4}$

Therefore, $\int_0^1 \tan^{-1} x\, dx = \int_0^{\pi/4} \theta \sec^2\theta\, d\theta$

$= \frac{\pi}{4} - \log\sqrt{2} = \frac{\pi}{4} - \frac{1}{2}\log 2$

Q.2 (2)

$I = \int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$

Put $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$ and $x = \sin t$

Also $t = 0$ to $\frac{\pi}{4}$ as $x = 0$ to $\frac{1}{\sqrt{2}}$

$\Rightarrow I = \int_0^{\pi/4} t.\sec^2 t\, dt = \frac{\pi}{4} - \frac{1}{2}\log 2$.

Q.3 (1)

Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$

Put $\sin x - \cos x = t$, then $(\sin x + \cos x)dx = dt$

$I = \int_{-1}^{0} \frac{dt}{9 + 16(1-t^2)} = \int_{-1}^{0} \frac{dt}{25 - 16t^2}$

$= \frac{1}{10}\int_{-1}^{0}\left(\frac{1}{5-4t} + \frac{1}{5+4t}\right)dt$

$= \left|\frac{1}{10}.\frac{1}{4}[\log(5+4t) - \log(5-4t)]\right|_{-1}^{0}$

$= \frac{1}{40}(\log 9 - \log 1) = \frac{1}{20}\log 3$.

Q.4 (2)

Let $I = \int_0^{\pi/6} \frac{\sin x}{\cos^3 x} dx = \int_0^{\pi/6} \tan x \sec^2 x\, dx$

Put $t = \tan x \Rightarrow dt = \sec^2 x\, dx,$ then we have

$I = \int_0^{\frac{1}{\sqrt{3}}} t\, dt = \left[\frac{t^2}{2}\right]_0^{\frac{1}{\sqrt{3}}} = \frac{1}{6}$

Q.5 (1)

$\int_{-\pi/4}^{\pi/4} e^{-x}\sin x\, dx = \left[\frac{e^{-x}}{2}(-\sin x - \cos x)\right]_{-\pi/4}^{\pi/2}$

$= \frac{1}{2}[e^{-x}(-\sin x - \cos x)]_{-\pi/4}^{\pi/2}$

$= \frac{1}{2}\left[e^{-\pi/2}(-1-0) - \left\{e^{\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right\}\right] = -\frac{e^{-\pi/2}}{2}$.

Q.6 (1)

$I = \int_0^1 \sqrt{\frac{1-x}{1+x}} dx = \int_0^1 \sqrt{\frac{1-x}{1+x}}.\frac{\sqrt{1-x}}{\sqrt{1-x}} dx$

$= \int_0^1 \frac{1-x}{\sqrt{1-x^2}} dx = \int_0^1 \frac{dx}{\sqrt{1-x^2}} - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$

$I = [\sin^{-1} x]_0^1 + [\sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - 1$

Q.7 (1)

$I = \int_1^x \frac{2\log x}{x} dx$

Let $\log x = t \Rightarrow \frac{dx}{x} = dt$

$\therefore I = 2\int_0^{\log x} t\, dt = 2\left[\frac{t^2}{2}\right]_0^{\log x} = (\log x)^2$.

Q.8 (1)

We have $\int_3^8 \frac{2-3x}{x\sqrt{1+x}} dx = I$

Put $1 + x = t^2 \Rightarrow dx = 2t\, dt$

When $x = 3 \to 8,$ then $t = 2 \to 3$

$\therefore I = 2\int_2^3 \frac{5-3t^2}{t^2-1} dt$; $I = 2\int_2^3 \left(\frac{2}{t^2-1} - 3\right) dt$

$$I = 2\left[\frac{2}{2.1}\log\frac{t-1}{t+1} - 3t\right]_2^3;\ I = 2\log\left(\frac{3}{2e^3}\right).$$

Q.9 (2)

$$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$

Since $\int_0^a xf(x)dx = \frac{1}{2}a\int_0^a f(x)dx,$

if $f(a-x) = f(x)$.

Q.10 (3)

$$I = \int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx \quad .....(i)$$

$$= \int_0^{\pi/2}\frac{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}dx$$

$$= \int_0^{\pi/2}\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx \quad .....(ii)$$

Now adding (i) and (ii), we get

$$2I = \int_0^{\pi/2}\frac{\sqrt{\cot x}+\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx = [x]_0^{\pi/2} \Rightarrow I = \frac{\pi}{4}.$$

Q.11 (2)

$$I = \int_0^{\pi} x\log\sin x\, dx \quad .....(i)$$

$$= \int_0^{\pi}(\pi - x)\log\sin(\pi - x)\, dx \quad .....(ii)$$

By adding (i) and (ii), we get

$$2I = \int_0^{\pi}\pi\log\sin x\, dx \Rightarrow I = \frac{2\pi}{2}\int_0^{\pi/2}\log\sin x\, dx$$

$$= \pi\left(\frac{\pi}{2}\log\frac{1}{2}\right) = \frac{\pi^2}{2}\log\frac{1}{2}$$

Q.12 (4)

$$\int_0^{\pi/2}\log\tan x\, dx = \int_0^{\pi/2}\log\left(\frac{\sin x}{\cos x}\right)dx$$

$$= \int_0^{\pi/2}\log\sin x\, dx - \int_0^{\pi/2}\log\cos x\, dx = 0$$

$$\left\{\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}.$$

Q.13 (1)

Since

$$f(-\theta) = \log\left(\frac{2-\sin\theta}{2+\sin\theta}\right)^{-1} = -\log\left(\frac{2-\sin\theta}{2+\sin\theta}\right) = -f(\theta)$$

$\therefore$ $f(x)$ is an odd function of x.

Therefore, $2\int_0^{\pi/2}\log\left(\frac{2-\sin\theta}{2+\sin\theta}\right)d\theta = 0$.

Q.14 (4)

$$I = \int_0^{2\pi}\frac{\sin 2\theta}{a - b\cos\theta}d\theta = \int_0^{2\pi}\frac{\sin(2\pi - 2\theta)}{a - b\cos(2\pi-\theta)}d\theta$$

$$\Rightarrow I = -\int_0^{2\pi}\frac{\sin 2\theta}{a-b\cos\theta}d\theta$$

$$\Rightarrow 2I = 0 \Rightarrow \int_0^{2\pi}\frac{\sin 2\theta}{a-b\cos\theta}d\theta = 0.$$

Q.15 (1)

$$I = \int_{-1/2}^{1/2}(\cos x)\left[\log\left(\frac{1-x}{1+x}\right)\right]dx \quad ...(i)$$

$$\therefore\ I = \int_{-1/2}^{1/2}\cos(-x)\left[\log\left(\frac{1+x}{1-x}\right)\right]dx$$

$$\Rightarrow I = -\int_{-1/2}^{1/2}\cos x\left[\log\left(\frac{1-x}{1+x}\right)\right]dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_{-1/2}^{1/2}\cos x\left[\log\left(\frac{1-x}{1+x}\right)\right]dx - \int_{-1/2}^{1/2}\cos x\left[\log\left(\frac{1-x}{1+x}\right)\right]dx$$

or $2I = 0$ or $I = 0$.

Q.16 (2)

Let $f(x) = \int_0^{\pi} e^{\sin^2 x}\cos^3(2n+1)x.dx$

Since $\cos(2n+1)(\pi - x) = \cos[(2n+1)\pi - (2n+1)x]$

$= -\cos(2n+1)x$ and $\sin^2(\pi - x) = \sin^2 x$

Hence by the property of definite integral,

$$\int_0^{\pi} e^{\sin^2 x}\cos^3(2n+1)x\, dx = 0,\ [f(2a-x) = -f(x)]$$

Q.17 (3)

$\int_{-3}^{3}\frac{x^2\sin x}{1+x^6}dx = 0$. By the property of definite integral,

$\int_{-a}^{a} f(x)dx = 0$, when $f(x) = -f(-x)$.

Q.18 (4)

$I = \int_0^{\pi/2} \frac{dx}{1+\tan^3 x} = \int_0^{\pi/2} \frac{\cos^3 x}{\sin^3 x + \cos x^3} dx$ .....(i)

$= \int_0^{\pi/2} \frac{\sin^3 x}{\cos^3 x + \sin^3 x} dx$ .....(ii)

Adding (i) and (ii), we get $2I = \int_0^{\pi/2} dx \Rightarrow I = \frac{\pi}{4}$.

Q.19 (2)

$I = \int_0^{\pi} \sqrt{(1-\cos 2x)}dx + \int_{\pi}^{2\pi} \sqrt{(1-\cos 2x)}dx + ...$

$..... + \int_{(r-1)\pi}^{r\pi} \sqrt{(1-\cos 2x)}dx + ... + \int_{99\pi}^{100\pi} \sqrt{(1-\cos 2x)}dx$

$\because \int_0^{na} f(x)dx = n\int_0^a f(x)dx$, if

$f(a+x) = f(x)$

$\therefore I = 100\int_0^{\pi} \sqrt{(1-\cos 2x)}dx$

$I = 100\sqrt{2}\int_0^{\pi} \sin x dx = 200\sqrt{2}\int_0^{\pi/2} \sin x \, dx$

$= 200\sqrt{2}[-\cos x]_0^{\pi/2} = 200\sqrt{2}$

Q.20 (3)

Let $I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$.

Then, $I = \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx$,

$\left[\because \int_0^{\pi/2} f(x)dx = \int_0^{\pi/2} f\left(\frac{\pi}{2} - x\right) dx\right]$

$\Rightarrow I = -\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx = -I$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

Q.21 (3)

$f(\cos x)$ is an even function.

$\because f(\cos(-x)) = f(\cos x)$

$\therefore \int_{-\pi/2}^{\pi/2} f(\cos x)dx = 2\int_0^{\pi/2} f(\cos x)dx = 2\int_0^{\pi/2} f(\sin x)dx$.

Q.22 (1)

(1) $I = \int_{-1/2}^{1/2} \cos x \ln\left(\frac{1+x}{1-x}\right) dx$

$\cos x \ln\left(\frac{1+x}{1-x}\right)$ is an odd function,

$(\because f(-x) = -f(x)$

$\therefore I = 0$

Q.23 (1)

Let $f(x) = \log(x + \sqrt{1+x^2})$

Now,

$f(-x) = \log\left(\sqrt{1+x^2} - x\right) = \log(\sqrt{1+x^2} - x).\frac{(\sqrt{1+x^2}+x)}{(\sqrt{1+x^2}+x)}$

$= \log\frac{[(1+x^2)-x^2]}{(\sqrt{1+x^2}+x)} = \log 1 - \log(\sqrt{1+x^2}+x)$

$= -\log(\sqrt{1+x^2}+x) = -f(x)$

Hence, $\int_{-1}^{1} \log(x+\sqrt{1+x^2}) = 0$,

$\left[\because \int_{-a}^{a} f(x) = 0, \text{if } f(-x) = -f(x)\right]$.

Q.24 (1)

$\int_0^9 [\sqrt{x}+2]dx = \int_0^1 2dx + \int_1^4 3dx + \int_4^9 4dx$

$= 2 + (12-3) + (36-16) = 2 + 9 + 20 = 31$

Q.25 (3)

$I = \int_0^{\pi/2} \sin 2x \log\tan x \, dx$,

$I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log\tan\left(\frac{\pi}{2} - x\right) dx$,

$[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx]$

$= \int_0^{\pi/2} \sin 2x \log\cot x \, dx = -\int_0^{\pi/2} \sin 2x \log\tan x \, dx$

$\therefore I = -I \Rightarrow 2I = 0 \Rightarrow I = 0.$

Q.26 (3)

Put $x = \sin\theta$, we get

$$\int_0^1 \frac{\log x}{\sqrt{1-x^2}}dx = \int_0^{\pi/2} \frac{\log \sin\theta.\cos\theta}{\cos\theta}d\theta$$

$$= \int_0^{\pi/2} \log \sin\theta\, d\theta = -\frac{\pi}{2}\log 2$$

Q.27 (1)

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad .....(i)$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}$$

.....(ii)

(Since $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Adding (i) and (ii), we get, $2I = \int_{\pi/6}^{\pi/3} dx$

$$I = \frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{12}.$$

Q.28 (2)

$$\lim_{n\to\infty} \frac{1^{99}+2^{99}+.....+n^{99}}{n^{100}} = \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r^{99}}{n^{100}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{99} = \int_0^1 x^{99}dx = \left[\frac{x^{100}}{100}\right]_0^1 = \frac{1}{100}.$$

Q.29 (2)

$$L = \lim_{n\to\infty}\sum_{r=1}^{2n}\frac{1}{n}.\frac{r/n}{\sqrt{1+(r/n)^2}}$$

$$= \int_0^2 \frac{x}{\sqrt{1+x^2}}dx = \sqrt{5}-1$$

Q.30 (4)

$$\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{2n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{n+n}\right]$$

$$= \frac{1}{n}\lim_{n\to\infty}\left[1+\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+....+\frac{1}{1+\frac{n}{n}}\right]$$

$$= \frac{1}{n}\lim_{n\to\infty}\sum_{r=0}^{n}\left[\frac{1}{1+\frac{r}{n}}\right] = \int_0^1 \frac{1}{1+x}dx$$

$$= [\log_e(1+x)]_0^1 = \log_e 2 - \log_e 1 = \log_e 2.$$

JEE-MAIN
OBJECTIVE QUESTIONS

Q.1 (1)

$$\int_1^x \frac{dx}{|t|\sqrt{t^2-1}} = \pi$$

$$\Rightarrow \left[\sec^{-1}\right]_1^x = \frac{\pi}{6} = \sec^{-1}x - \sec^{-1}1$$

$$= \frac{\pi}{6} = \sec^{-1}x = \frac{\pi}{6}$$

$$\Rightarrow x = \sec^{-1}\frac{\pi}{6} = \frac{2}{\sqrt{3}}$$

Q.2 (3)

$$I = \int_0^1 \frac{dx}{x^2+2x\cos\alpha+\sin^2\alpha+\cos^2\alpha}$$

$$= \int_0^1 \frac{dx}{(x+\cos\alpha)^2+\sin^2\alpha}$$

$$= \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{x+\cos\alpha}{\sin\alpha}\right)\right]_0^1 = \frac{\alpha}{2\sin\alpha}$$

Q.3 (3)

$$\int_0^2 x^2 f(x)dx = \int_0^1 x^3 dx + \int_1^2 (x^3-x^2)dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4}-\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{4}+\left[4-\frac{8}{3}-\left(\frac{1}{4}-\frac{1}{3}\right)\right] = 4-\frac{8}{3}+\frac{1}{3} = 4-\frac{1}{3} = \frac{5}{3}$$

Q.4 (3)

$$\int_n^{n+1} f(x)dx = n^2$$

$$\int_{-2}^{4} f(x)dx = \int_{-2}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx +$$

$$\int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_3^4 f(x)dx$$

= 4 + 1 + 0 + 1 + 4 + 9 = 19

Q.5 (4)

$$I = \int_0^{\pi} |1+2\cos x|\,dx$$

$$= \int_0^{2\pi/3}(1+2\cos x)dx - \int_{2\pi/3}^{\pi}(1+2\cos x)dx$$

$$= (x+2\sec x)|_0^{2\pi/3} - (x+2\sec x)|_{2\pi/3}^{\pi}$$

$$= \frac{\pi}{3} + 2\sqrt{3}$$

Q.6 (1)

$$I = \int_{-1}^{3}(|x-2|+[x])dx$$

$$= \int_{-1}^{2}|x-2|+dx + \int_2^3 |x-2|+dx + \int_{-1}^{0}(-1)dx +$$

$$\int_0^1 0.dx + \int_1^2 1.dx + \int_2^3 2.dx$$

$$= \int_{-1}^{2}(2-x)\,dx + \int_2^3 (x-2)dx \; -1 + 0 + 1 + 2$$

$$= 2x - \frac{x^2}{2}\Big|_{-1}^{2} + \frac{x^2-2x}{2}\Big|_2^3 + 2 = 7$$

Q.7 (3)

$$I = \int_0^1 \underset{\text{I}}{x}\,\underset{\text{II}}{f''(2x)}dx = \left[\frac{x f'(2x)}{2} - \frac{f(2x)}{4}\right]_0^1$$

$$= \frac{f'(2)}{2} - \frac{f(2)}{4} + \frac{f(0)}{4} = \frac{5}{2} - \frac{3}{4} + \frac{1}{4} = 2$$

Q.8 (1)

$$\int_{-1}^{3/2}|x\sin\pi x|\,dx$$

$$= \int_{-1}^{1}|x\sin\pi x|\,dx + \int_1^{3/2}|x\sin\pi x|\,dx$$

$$= 2\int_0^1 |x\sin\pi x|\,dx + \int_1^{3/2}|x\sin\pi x|\,dx$$

$$= 2\int_0^1 x\sin\pi x\,dx - \int_1^{3/2} x\sin\pi x\,dx$$

$$\int x\cos\pi x dx = -\frac{x\cos\pi x}{\pi} + \frac{\sin\pi x}{\pi^2}$$

$$\int_0^1 x\sin\pi x dx = \frac{1}{\pi}$$

$$\int_1^{3/2} x\sin\pi x\,dx = \frac{-1}{\pi^2} - \frac{1}{\pi}$$

$$I = \frac{2}{\pi} + \frac{2}{\pi^2} + \frac{1}{\pi} = \frac{3\pi+1}{\pi^2}$$

k = 3π + 1

Q.9 (2)

$$I = \int_0^{\pi/4}\frac{x\sin x}{\cos^3 x}dx = \int_0^{\pi/4} x\tan x\sec^2 x\,dx$$

$$= x\,\frac{\tan^2 x}{2}\Big|_0^{\pi/4} - \int \frac{\tan^2 x}{2}dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\int(\sec^2 x - 1)\,dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[\tan x - x\right]_0^{\pi/4} = \frac{\pi}{4} - \frac{1}{2}$$

Q.10 (1)

$$I = \int_{\log\pi-\log 2}^{\log\pi}\frac{e^x}{1-\cos\left(\frac{2}{3}e^x\right)}dx$$

Put $\frac{e^x}{3} = t \Rightarrow e^x\,dx = 3dt$

$$= 3\int_{\pi/6}^{\pi/3}\frac{dt}{1-\cos 2t} = \frac{3}{2}\int_{\pi/4}^{\pi/3}\frac{dt}{\sin^2 t} = \frac{3}{2}\int \operatorname{cosec}^2 t\,dt$$

$$= -\frac{3}{2}\ [\cot t]_{\pi/6}^{\pi/3} = -\frac{3}{2}\left[\frac{1}{\sqrt{3}} - \sqrt{3}\right] = \sqrt{3}$$

Q.11 (1)

$$I_1 = \int_{e}^{e^2} \frac{dx}{\ell nx} \quad ; \qquad I_2 = \int_1^2 \frac{e}{x}\,dx$$

Put $\ell nx = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$$I_1 = \int_1^2 \frac{e^t}{x}\,dt = I_2$$

Q.12 (1)

$$\int_0^{\pi/2} \ell n|\tan x + \cot x|\,dx$$

$$= \int_0^{\pi/2} \ell n\left(\frac{2}{\sin 2x}\right)dx = \int_0^{\pi/2} \ell n2dx - \int_0^{\pi/2} \ell n(\sin 2x)dx$$

Q.13 (4)

$$I = \int_0^{\left(\frac{\pi}{2}\right)^{1/3}} x^5 \sin x^3 dx$$

$$x^3 = t \Rightarrow x^2 dx = \frac{dt}{3}$$

$$I = \frac{1}{3}\int_0^{\pi/2} t\sin t\,dt = \frac{1}{3}\ [-t \cos t + \int \cos t\,dt\,]$$

$$= \frac{1}{3}[-t \cos t + \sin t]_0^{\pi/2} = \frac{1}{3}$$

Q.14 (2)

$$I = \int_0^{\infty} [2e^{-x}]\,dx$$

Let $2e^{-x} = t$

$$-2e^{-x}\,dx = dt \Rightarrow dx = \frac{-dt}{t}$$

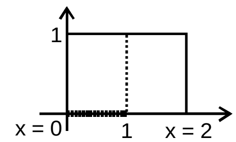
$$= -\int_2^0 [t]\ \frac{dt}{t} = \int_0^2 [t]\frac{dt}{t} = \int_0^1 \frac{0}{t}dt + \int_1^2 \frac{1}{t}dt = \ell n2$$

Q.15 (3)

$f(x) \quad = 0$ where $\pi = \frac{n}{n+1}$, n = 1, 2, 3, ...

$= 1$ else where

Find $= \int_0^2 f(x)dx = 2 \times 1 = 2$



Q.16 (4)

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\infty} e^{-ax^2} dx \quad \text{Put } \sqrt{ax} = t$$

$$dx = \frac{dt}{\sqrt{a}} = \frac{1}{\sqrt{a}}\int_0^{\infty} e^{-t^2}\ dt = \frac{\sqrt{3}}{2\sqrt{a}} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

Q.17 (2)

$$A = \int_0^1 \frac{e^t dt}{1+t}$$

$$I = \int_{a-1}^{a} \frac{e^{-t}}{t-a-1}\,dt = -\int_{a-1}^{a} \frac{e^{-t}}{1+a-t}dt$$

Put $a - t = z$

$dt = -dz$

$$= \int_1^0 \frac{e^{a(z-a)}dz}{1+z} = -\int_0^1 \frac{e^z.dz}{1+z} = -Ae^{-a}$$

Q.18 (3)

$$I = \int_1^2 ([x^2] - [x]^2)\,dx$$

$$= \int_1^{\sqrt{2}} 1.dx + \int_{\sqrt{2}}^{\sqrt{3}} 2.dx + \int_{\sqrt{3}}^{2} 3.dx - \int_1^2 1.dx$$

$= (\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3\,(2 - \sqrt{3}) - 1$

$= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3} - 1$

$4 - \sqrt{2} - \sqrt{3}$

Q.19 (2)

$$I = \int_0^{\pi} f(x)\sin x\,dx + \int_0^{\pi} f''(x)\sin x\,dx$$

$$= -\cos x + f(x)\,|_0^{\pi} + \int_0^{\pi} f'(x)\cos x\,dx + \sin x\,f'(x)|_0^{\pi}$$

$$-\int_0^{\pi} \cos f'(x)\,dx$$

I = f(x) + f(0)
5 = f(x) + f(0)
5 = 2 + f(0)
f(0) = 3

Q.20 (1)

$$\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2}$$

$$\sec^{-1}x - \sec^{-1}\sqrt{2} = \frac{\pi}{2}$$

$$\sec^{-1}x = \frac{\pi}{2} = \frac{\pi}{2}$$

$$\sec^{-1}x = \frac{\pi}{2} + \frac{\pi}{4}$$

$$\sec^{-1}x = \frac{3\pi}{4}$$

$$x = \sec\frac{3\pi}{4}$$

$$x = -\sqrt{2}$$

Q.21 (3)

$$I = \int_0^{\pi/2} \frac{(\sin x+\cos x)^2}{|\sin x+\cos x|}\,dx = \int_0^{\pi/2} \frac{(\sin x+\cos x)^2}{(\sin x+\cos x)}\,dx$$

$$= \int_0^{\pi/2} (\sin x+\cos x)\,dx \quad [\sin x - \cos x]_0^{\pi/2} = 1 + 1 = 2$$

Q.22 (3)

Let $I_1 = \int_a^b f(x)g(x)dx$ [given $\frac{d}{dx}f(x) = g(x)$]

$$\Rightarrow I_1 = \left[f(x)\int g(x)dx\right]_a^b - \int_a^b\left(\frac{d}{dx}f(x)\int g(x)dx\right)dx$$

$$= \left[f^2(x)\right]_a^b - \int_a^b f(x)g(x)dx \Rightarrow 2I_1 = [f(b)]^2 - [f(a)]^2$$

$$\Rightarrow I_1 = \frac{[f(b)]^2 - [f(a)]^2}{2}$$

Q.23 (2)

$$I = \int_1^5 [\,|x-3|\,]\,dx$$

$$= \int_{-2}^{2} [\,|x|\,]\,dx = 2\int_0^2 [\,|x|\,]\,dx$$

$$= 2\int_0^2 [x]\,dx = 2[1] = 2$$

Q.24 (1)

$$\int_1^2 (x - \log_2 a)\,dx = 2\log_2\left(\frac{2}{a}\right)$$

$$\left.\frac{x^2}{2} - (\log_2 a)x\right|_a^2 = 2\log_2\left(\frac{2}{a}\right)$$

$$2 - 2\log_2 a = 2\log_2\frac{2}{a}$$

$2 - 2\log_2 a = 2\log_2 2 - 2\log_2 a$
1 = 1
a > 0 because of log properties.

Q.25 (2)

$$I = \frac{1}{C}\cdot\int_{ac}^{bc} f\left(\frac{x}{c}\right)dx$$

Put $\frac{x}{c} = t \Rightarrow \frac{dx}{c} = dt = \int_a^b f(t)\,dt$

Q.26 (3)

$$\int_{\ell n2}^{x} \frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$$

Put $e^x - 1 = t^2$
$e^x\,dx = 2t\,dt$

$$dx = \frac{2t\,dt}{(t^2+1)}$$

$$2\int_1^{\sqrt{e^x-1}} \frac{dt}{t^2+1} = \frac{\pi}{6}$$

$$2\tan^{-1}\sqrt{e^x-1} = 30^{\circ} + 90^{\circ}$$

$\tan^{-1}\sqrt{e^x - 1} = 60^o$

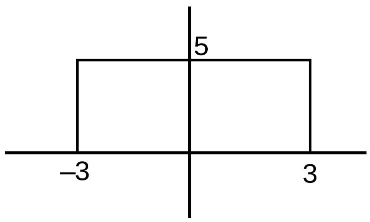
$\sqrt{e^x - 1} = \sqrt{3}$

$e^x - 1 = 3$

$e^x = 4 \Rightarrow x = \ell n\ 4$

Q.27 (4)

$$\int_{-3}^{3} f(x)\,dx = 5 \times (3 + 3) = 30$$



Q.28 (1)

$$I = \int_{5/2}^{5} \frac{\sqrt{(25 - x^2)^3}}{x^4}\,dx$$

Put $x = 5\sin\theta$

$\Rightarrow \quad dx = 5\cos\theta\, d\theta$

$$= \int_{\pi/6}^{\pi/2} \cot^4\theta\, d\theta = \int_{\pi/6}^{\pi/2} \cot^2\theta\,(\mathrm{cosec}^2\theta - 1)\,d\theta$$

$$= \int \cot^2\theta\,\mathrm{cosec}^2\theta\, d\theta - \int \cot^2\theta\, d\theta$$

$$= \left.\frac{\cot^2 3\theta}{3}\right|_{\pi/6}^{\pi/2} - \int \mathrm{cosec}^2\theta + \theta$$

$$= -\frac{\cot^3\theta}{3} + \cot\theta + \theta\,\Big|_{\pi/6}^{\pi/2}$$

$$= \frac{\pi}{2} + \frac{(\sqrt{3})^3}{3} - \sqrt{3} - \frac{\pi}{6} = \frac{\pi}{3}$$

Q.29 (2)

$$\int_{-1}^{1} f(x)\,dx = 2\,[A_1 + A_2 + A_3] = 2$$

$$\left[\left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{4}\right) + \left(\frac{1}{2}\times\frac{1}{4}\right) + \left(\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4}\right)\right] = \frac{3}{8}$$

Q.30 (3)

$$I = \int_{\ell n\lambda}^{-\ell n\lambda} \frac{f\left(\frac{x^2}{4}\right)[f(x) - f(-x)]}{f\left(\frac{x^2}{4}\right)[g(x) + g(-x)]}\,dx$$

odd function by P – 5

$I = 0$

Q.31 (4)

$$2I = \int_{2-\log 3}^{3+\log 3} \frac{\log(4+x)}{\log(4+x) + \log(9-x)}\,dx$$

using king

replace x by 5 – x

$$I = \int_{2-\log 3}^{3+\log 3} \frac{\log(9-x)}{\log(9-x) + \log(4+x)}\,dx$$

$$2I = \int_{2-\log 3}^{3+\log 3} dx \Rightarrow I = \frac{1}{2} + \log 3$$

Q.32 (3)

$$I_1 = \int_0^{3\pi} f(\cos^2 x)\,dx \quad ; \text{ period is } \pi$$

$$= 3\int_0^{\pi} f(\cos^2 x)\,dx$$

$I_1 = 3I_3$

Similalry $I_2 = 2I_3$

$I_2 + I_3 \ \ 3I_3$

$I_2 + I_3 = I_1$

Q.33 (3)

$$I = \int_0^{11} \frac{11^x}{11^{[x]}}\,dx = \frac{k}{\log 11}$$

$$I = \int_0^{11} \frac{11^x}{11^{[x]}}\,dx = \int_0^{11} \frac{11^x}{11^{x-\{x\}}}\,dx$$

$$= \int_0^{11} 11^{\{x\}}\,dx = 11\int_0^{1} 11^x\,dx = \frac{11}{\log 11}\,[11^x]_0^1 = \frac{110}{\log 11}$$

$k = 110$

Q.34 (4)

$f(x) = \frac{-1}{x^2} f\left(\frac{1}{x}\right)$

$I = \int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)dx = -\int_{\sin\theta}^{\operatorname{cosec}\theta} \frac{1}{x^2} f\left(\frac{1}{x}\right)dx$

$\frac{1}{x} = t \Rightarrow -\frac{1}{x^2}dx = dt$

$I = \int_{\operatorname{cosec}\theta}^{\sin\theta} f(t)\,dt = -\int_{\sin\theta}^{\operatorname{cosec}\theta} f(t)\,dt$

$2I = 0 \Rightarrow I = 0$

Q.35 (2)

$f(x) = f(a - x)$

$g(x) + g(a - x) = 2$

$I = \int_0^a f(x)\,g(x)\,dx = \int_0^a f(a-x)g(a-x)dx$

$I = \int_0^a f(x)\ (2 - g(x))\ dx = 2\int_0^a f(x)dx - I$

$2I = 2\int_0^a f(x)dx \Rightarrow I = \int_0^a f(x)dx$

Q.36 (3)

$I = \int_4^{10} \frac{[x^2]}{[(x-14)]^2 + [x^2]} dx$

$I = \int_4^{10} \frac{[x-14]^2}{[(x-14)^2]+[x^2]} dx$

$2I = \int_4^{10} 1.dx \Rightarrow I = 3$

Q.37 (2)

$I = \sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)dx\right)$

$= \int_0^1 f(x)dx + \int_0^1 f(1+x)dx + \int_0^1 f(2+x)dx + ... + \int_0^1 f(99+x)dx$

$= \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^3 f(x)dx + ... + \int_{99}^{100} f(x)dx$

$= \int_0^{100} f(x)dx = 1 = a$

Q.38 (2)

$\int_0^{n/2} \sin[2x]\,dx$ $2x = t$

$2dx = dt$

$dx = \frac{dt}{2}$

$\frac{1}{2}\int_0^n \sin[t]\,dt$

$= \frac{1}{2}\left[\int_0^1 \sin 0\ dt + \int_1^2 \sin 1\,dt + \int_2^3 \sin 2\,dt + \int_3^\pi \sin 3\,dt\right]$

$= \frac{1}{2}$ [sin 1 + sin 2 + (π – 3) sin 3]

Q.39 (2)

$I = \int_{-\pi/2}^{\pi/2} \frac{|x|\,dx}{8\cos^2 2x + 1} = 2\int_0^{\pi/2} \frac{|x|\,dx}{8\cos^2 2x+1}$

$I = 2\int_0^{\pi/2} \frac{x\,dx}{8\cos^2 2x+1}$

$I = 2\int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)dx}{8\cos^2 2x+1}$

$2I = \pi\int_0^{\pi/2} \frac{dx}{8\cos^2 2x+1}$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$

$2I = \frac{\pi}{2}\int_0^{\pi} \frac{dt}{8\cos^2 t+1}$

$2I = \pi\int_{\pi/2}^{\pi/2} \frac{-dt}{8\cos^2 t+1}$

$I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sec^2 t\,dt}{a+\tan^2 t} = \frac{3}{2}\left(\frac{1}{3}\right)\tan^{-1}\tan t\,|_0^{\pi/2} = \frac{\pi^2}{12}$

Q.40 (1)

$$I = \int_0^{\pi} x f(\sin x)\,dx = \int_0^{\pi} (\pi - x) f(\sin x)\,dx$$

$$= \pi \int_0^{\pi/2} f(\sin x)\,dx$$

Q.41 (4)

$$I = \int_0^{2n\pi} |\sin x|_{sec} - \int_0^{2n\pi} \left[\left|\frac{\sin x}{2}\right|\right]_{dx}^{a}$$

$$= \int_0^{2n\pi} |\sin x|\,dx \quad -1 \le \sin x \le 1$$

$$= 2n \int_0^{\pi} |\sin x|\,dx \quad -\frac{1}{2} \le \frac{\sin x}{2} \le \frac{1}{2}$$

$= 2n\ (2) = 4n$

Q.42 (4)

$$\int_0^{\pi/3} f(x)\,dx = 0 \int_0^{\pi/4} \tan x\,dx + \int_{\pi/4}^{\pi/3} \cot x\,dx$$

$$= \ell n \sec x|_0^{\pi/4} + \ell n \sec x|_{\pi/4}^{\pi/3}$$

$$= \ell n\sqrt{2} + \ell n\,\frac{\sqrt{3}}{2} - \ell n\,\frac{1}{\sqrt{2}} = \ell n\sqrt{3}$$

Q.43 (2)

Q.44 (4)

Q.45 (1)

$$f(x) = f(x) + f\left(\frac{1}{x}\right)$$

$$f(x) = \int_4^{x} \frac{\log t}{1+t}\,dt$$

$$f\left(\frac{1}{x}\right) = \int_4^{1/x} \frac{\log t}{1+t}\,dt$$

$$t = \frac{1}{4}$$

$$\Rightarrow\ dt = -\frac{du}{u^2}$$

$$= -\int_4^{u} \frac{\ell n - \frac{1}{4}}{1+\frac{1}{4}} \cdot \frac{du}{u^2} = \int_1^{x} \frac{\ell n 4}{u(1+u)}\,du$$

$$f\left(\frac{1}{x}\right) \int_1^{x} \frac{\ell n t}{(1+t)t}\,dt$$

$$f(x) = f(x) + f\left(\frac{1}{x}\right) = \int_1^{x} \frac{\ln t}{(t+1)} \left(\frac{1+1}{t}\right) dt$$

$$f(x) = \int_1^{x} \frac{\ell n t}{t}\,dt = \left.\frac{\ell n^2 t}{2}\right|_1^{x}$$

$$f(x) = \frac{\ell n^2 x}{2} \Rightarrow f(e) = \frac{\ell n^2 e}{2} = \frac{1}{2}$$

Q.46 (2)

$f(a + b - x) = f(x)$

$$I = \int_a^{a} x f(x)\,dx$$

King

$$I = \int_a^{b} (a+b-x) + (a + b - x)\,dx$$

$$I = \int_a^{b} (a+b-x) f(n)\,dx$$

$$2I = (a + b) \int_a^{b} f(x)\,dx$$

Q.47 (3)

$$\because\ f'(x) = f(x) \Rightarrow \frac{f'(x)}{f(x)} = 1$$

$$\int \frac{f'(x)}{f(x)}\,dx = \int dx \Rightarrow \ell n\, f(x) = x + c$$

$f(x) = e^{x + c}$.....(1)

$f(0) = 1 \Rightarrow e^c = 1 \Rightarrow c = 0$

Now $f(x) = e^x$

$f(x) + g(x) = x^2 \Rightarrow g(x) = x^2 - e^x$

$$I = \int_0^1 f(x)\,g(x)\,dx = \int_0^1 e^x\,\{x^2 - e^x\}\,dx$$

$$= e - \frac{e^2}{2} - \frac{3}{2}$$

Q.48 (4)

$$I = \int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$$

Applying King

$$I = \int_0^{\pi/2} \frac{dx}{1+\cot^3 x}$$

Add

$$2I = \int_0^{\pi/2} dx$$

$$I = \frac{\pi}{4}$$

Q.49 (1)

$$I = \int_{-1}^{3} \left(\tan^{-1}\frac{x}{1+x^2} + \tan^{-1}\frac{x^2+1}{x}\right) dx$$

$$I = \int_{-1}^{1} \left(\tan^{-1}\frac{x}{1+x^2} + \tan^{-1}\frac{x^2+1}{x}\right) dx$$

O is odd function

$$+ \int_{1}^{3} \left(\tan^{-1}\frac{x}{1+x^2} + \tan^{-1}\frac{x^2+1}{}\right) dx$$

$$= \int_{1}^{3} \left(\tan^{-1}\frac{x}{1+x^2} + \cot^{-1}\frac{x}{1+x^2}\right) dx$$

$$= \int_{1}^{3} \frac{\pi}{2}.dx = \frac{\pi}{2}(z) = \pi$$

Q.50 (3)

$$I = \int_{-1}^{1} \frac{x^4}{1+e^{x^7}} dx$$

King Replace $x \to -x$

$$I = \int_{-1}^{1} \frac{x^4}{1+e^{-x^7}} dx$$

$$2I = \int_{-1}^{1} \frac{x^4(1+e^{x^7})}{(1+e^{x^7})} dx$$

$$I = \frac{1}{5}$$

Q.51 (3)

$$I = \int_{-1}^{1} \frac{\sin x + x^2}{3-|x|} dx = \underbrace{\int_{-1}^{1} \frac{\sin x + x^2}{3-|x|}}_{\text{O as odd Function}} + \int_{-1}^{1} \frac{x^2}{3-|x|} dx$$

$$= 2\int_0^1 \frac{x^2}{3-|x|} dx$$

Q.52 (4)

$$I = \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{r^3}{r^4+n^4}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{r^3}{n^4\left(1+\frac{r}{n}\right)^4}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{\left(\frac{r}{n}\right)^3}{1+\left(\frac{r}{n}\right)^4} \frac{1}{n}\right) = \int_0^1 \frac{x^3}{1+x^4} dx$$

$$= \frac{1}{4} \ell n \ (1+x^4)\Big|_0^1 = \frac{1}{4} \ell n \ 2$$

Q.53 (2)

$$\lim_{n\to\infty} \sum_{r=2n+1}^{3n} \frac{n}{r^2-n^2}$$

$$= \lim_{n\to\infty} \sum_{r=2n+1}^{3n} \frac{1}{\left(\left(\frac{r}{n}\right)^2-1\right)} \cdot \frac{1}{n}$$

$$= \int_2^3 \frac{dx}{x^2-1} = \frac{1}{2} \ \ell n \ \frac{x-1}{x+1}\Big|_2^3 = \ell n \ \sqrt{\frac{3}{2}}$$

Q.54 (3)

$$L = \lim_{h\to\infty} \left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{h^2}\right)...\left(1+\frac{n^2}{h^2}\right)\right]^{1/n}$$

$$\ell n L = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \ell n\left(1+\left(\frac{r}{n}\right)^2\right)$$

$$= \int_0^1 \ell n(1+x^2)dx$$

$$= x\ell n\,(1 + x^2) - 2x + 2\tan^{-1}x\,|_0^1$$

$$\ell nL = \ell n2 - 2 + \frac{\pi}{2} \Rightarrow L = \frac{2}{e^2}\; e^{\pi/2}$$

Q.55 (1)

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n}\left(\frac{r}{n}\right) \sec^2 \left(\frac{r}{n}\right)^2 = \int_0^1 x \sec^2 x^2\, dx$$

Put $x^2 = t$

$$x\,dx = \frac{dt}{2} = \frac{1}{2}\int_0^1 \sec^2 t = \frac{1}{2}\,[\tan t]_0^1 = \frac{1}{2}\tan 1$$

Q.56 (3)

$$I = \lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{\pi}{n} \sin\frac{r\pi}{n} = \pi \int_0^1 \sin \pi x\, dx = \pi$$

$$\left[-\frac{\cos \pi x}{\pi}\right]_0^1 = [-\cos\pi + 1] = 2$$

Q.57 (4)

$$f(x) = 1 + x + \int_1^x (\ell n^2 t + 2\ell nt)\,dt$$

$f'(x) = 1 + \ell n^2 x + 2\ell nx$

$f'(x) = 0$

$(\ell nx + 1)^2 = 0 \Rightarrow x = e^{-1}$

$$f\left(\frac{1}{e}\right) = 1 + \frac{1}{e} + \int_1^{1/e}\left[\underset{\uparrow\atop f(t)}{\ell n^2 t} + \underset{\uparrow\atop f'(t)}{\left(\frac{2}{t}\ell nt\right)}\right]dt$$

$$= 1 + \frac{1}{e} + t\;\ell n^2 t\,|_1^{1/e} = 1 + \frac{2}{e} = 1 + 2e^{-1}$$

Q.58 (1)

$f(x) = e^{g(x)}$

$$g(x) = \int_2^x \frac{t\,dt}{1+t^4}$$

$$g'(x) = \frac{x}{1+x^4}$$

$$g'(2) = \frac{2}{17}$$

$f'(x) = e^{g(x)} . g'(x)$

$f'(2) = e^{g(x)} . g'(2)$

$$= e^0 . \frac{2}{17} = \frac{2}{17}$$

Q.59 (1)

$$\lim_{x\to 2} \frac{\int_6^{f(x)} 4t^3}{x-2}\,dt$$

Apply L′Hospital

$$\lim_{x\to 2} \frac{4f^3(x).f'(x)}{1}$$

$$4f^3(2) . f^1(2) = 4 \times 6^3 \times \frac{1}{48} = 18$$

Q.60 (2)

$$I = \underset{h\to 0}{\text{Limit}} \frac{\int_a^x \ell n^2 t dt + \int_x^{x+h} \ell n^2 t dx - \int_a^x \ell n^2 t dt}{h}$$

$$\Rightarrow I = \underset{h\to 0}{\text{Limit}} \frac{\int_x^{x+h} \ell n^2 t dx}{h}$$

Using L hospital we get

$$I = \underset{h\to 0}{\text{Limit}}\; \ell n^2 (x+h) = \ell n^2 x$$

Q.61 (3)

$$L = \lim_{x\to 0} \frac{\int_0^{x^2} \sec^2 t\, dt}{x \sin x} = \lim_{x\to 0} \frac{\int_0^{x^2} \sec^2 t\, dt}{x^2}$$

$$= \lim_{x\to 0} \frac{\sec^2 x^2 . 2x}{2x} = \sec^2 (0) = 1$$

Q.62 (4)

$$\int_1^4 \frac{3}{x} e^{\sin x^3}\, dx = \int_1^4 \frac{3x^2}{x^3} e^{\sin x^3} dx$$

$x^3 = t$

$$3x^2\, dx = dt = \int_1^{64} \frac{e^{\sin t}}{t} dt = f(t)\,|_1^{64} = f(64) - f(1)$$

Q.63 (B)

$$I = \lim_{x\to 0} \frac{\int_0^{x^2} \cos t^2\, dt}{x \sin x} = \lim_{x\to 0} \frac{\int_0^{x^2} \cot^2 dt}{x^2}$$

$$= \lim_{x\to 0} \frac{\cos x^4 . 2x}{2x} = 1$$

Q.64 (2)

$$\int_a^y \cos t^2 \; dt = \int_a^{x^2} \frac{\sin t}{t} dt$$

differentiating both sides w.r.t x we get

$$\frac{d}{dx}\int_a^y \cot\, t^2\, dt = \frac{d}{dx}\int_a^{x^2} \frac{\sin t}{t} dt$$

$$\text{RHS} = \frac{\sin[x^2]}{x^2}\frac{dx^2}{dx} = 2 \times \frac{\sin x^2}{x^2}$$

L.H.S. = $\frac{d}{dy}$

$$\left(\int_a^y \cos t^2\, dt\right)\frac{dy}{dx} = \cos\, y^2 \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2 \sin x^2}{x \cos y^2}$$

Q.65 (2)

$$I_1 = \int_1^2 \frac{dx}{\sqrt{1+x^2}} = \ell n \left(x + \sqrt{x^2+1}\right)_1^2 = \ell n \left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)$$

$$I_2 = \int_1^2 \frac{dx}{x}$$

$= \ell n\ 2$

$I_2 > I_1$

Q.66 (1)

$$I = \int_0^1 c_2x^2 + c_1x + c_0 = \frac{c_2x^2}{3} + \frac{c_1x^2}{2} + c_0x \Big|_0^1$$

$$= \frac{c_2}{3} + \frac{c_1}{2} + c_0$$

I = 0 than definately one root will lie in (0, 1)

JEE-ADVANCED OBJECTIVE QUESTIONS

Q.1 **(A)**

$$I = \int_1^a [x] f'(x) dx$$

$$I = \int_1^2 1.f'(x)dx + \int_2^3 2.f'(x)dx + \int_3^4 3.f'(x)dx + \ldots$$

$$+ \int_{[a]}^a [a].f'(x)dx$$

= [f(2) – f(1)] + 2 {f(3) – f(2)} + 3 (f(4) – f(9)) + ... + [a] (f(a) – f[a]

= –f(1) – f(2) – f(3) – f(4) ... f(a) + [a] + (a)

= [a] f(a) – {f(1) + f(2) + f(3) ..... – f [a]}

Q.2 (D)

Q.3 (B)

$$I = \int_0^{\pi/4} \sin(x-[x])\, dx(x-[x])$$

$$0 < x < \frac{\pi}{4}$$

[x] = 0

$$= \int_1^3 [3-x]dx + \int_3^5 [x-3]dx$$

3 – x = t x – 3 = 4

dx = – dt dx = du

$$= -\int_2^0 [t]dt + \int_2^2 [u]du$$

$$= 2\int_0^2 [t]dt = 2\left[\int_0^1 0.dt + \int_0^2 0.dt\right] = 1 - \frac{1}{\sqrt{2}}$$

Q.4 (C)

$$I_1 = \int_0^1 \frac{e^x dx}{1+x} \backslash$$

$$I_2 = \int_0^1 \frac{x^2 dx}{e^{x^3}(2-x^3)} dx$$

Put $x^3 = t$

$x^2\,dx = \frac{dt}{3} \quad = \frac{1}{3}\int_0^1 \frac{dt}{e^t(2-5)}$

$1 - t = z$
$dt = -\,dz$

$= -\frac{1}{3}\int_1^0 \frac{dz}{e^{(1-z)}(1+z)} = \frac{1}{3}\int_0^1 \frac{e^x dz}{e^{(1-z)}(1+z)}$

$I_2 = \frac{1}{3e}\; I_1$

$\frac{I_1}{I_2} = 3e$

Q.5 (B)

$0 < x < \frac{\pi}{2}$

$I = \int_{1/\sqrt{2}}^{1/2} \cot x\; d(\cos x)$

$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x}\sin x\,dx \quad = -\;[\sin x]_{\pi/4}^{\pi/3}$

$= -\left[\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right] = -\left(\frac{\sqrt{3}-\sqrt{2}}{2}\right) = \frac{\sqrt{2}-\sqrt{3}}{2}$

Q.6 (C)

$I = \int_0^{\pi/3} [\sqrt{3}\,\tan x]\,dx$

$\sqrt{3}\;\tan x \;= t$

$\sqrt{3}\,\sec^2 x\;dx = dt$

$dx = \frac{dt}{\sqrt{3}(1+\tan^2 x)} = \frac{dt}{\sqrt{3}\left(1+\frac{t^2}{3}\right)} = \frac{\sqrt{3}dt}{t^2+3}$

$= \int_0^3 [t]\cdot\frac{\sqrt{3}dt}{t^2+3}$

$= \sqrt{3}\int_0^1 \frac{0.dt}{t^2+3} + \sqrt{3}\int_1^2 \frac{dt}{t^2+3} + 2\sqrt{3}\int_0^3 \frac{dt}{t^2+3}$

$= \tan^{-1}\left.\frac{t}{\sqrt{3}}\right|_1^2 + 2\,\tan^{-1}\left.\frac{t}{\sqrt{3}}\right|_2^3$

$= \tan^{-1}\frac{2}{\sqrt{3}} - \frac{\pi}{6} + 2\,\tan^{-1}\sqrt{3} - 2\,\tan^{-1}\frac{2}{\sqrt{3}}$

$= \frac{2\pi}{3} - \frac{\pi}{6} - \tan^{-1}\frac{2}{\sqrt{3}} = \frac{\pi}{2} - \tan^{-1}\frac{2}{\sqrt{3}}$

Q.7 (A)

$I = \int_0^{[x]} \{x\}\,dx = [x]\int_0^1 \{x\}\,dx = [x]\int_0^1 x\,dx = \frac{[x]}{2}$

Q.8 (D)

$I = \int_0^1 e^{2x-[2x]} d(x-[x]) = \int_0^1 e^{[2x]}dx$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$

$= \frac{1}{2}\int_0^2 e^{\{t\}}dt = \int_0^1 e^{\{t\}}dt = \int_0^1 e^t dt = e - 1$

Q.9 (D)

$I = \int_{\pi/4}^{\pi/3} \operatorname{cosec} x\; d(\sin x)$

$= \int_{\sin^{-1}\frac{\pi}{4}}^{\sin^{-1}} \cot x\,dx = \ell n\,\sin\,x \;\; \ell n \sin x\,|_{\sin^{-1} n/4}^{\sin^{-1}\pi/3}$

$= \ell n\,\sin\left(\sin^{-1}\frac{\pi}{3}\right) - \ell n\,\sin\left(\sin^{-1}\frac{\pi}{4}\right)$

$= \ell n\,\frac{\pi}{3} - \ell n\,\frac{\pi}{4} = \ell n\frac{4}{3}$

Q.10 (B)

$I = \int_{-3\pi/2}^{-\pi/2} \{(x+\pi)^3 + \cos^2(x+3\pi)\}\,dx$

Put $x + x = t \quad \Rightarrow \quad dx = d\,t$

$I = \int_{-\pi/2}^{\pi/2} (t^3 + \cos^2 t)\,dt$

$I = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

Q.11 (A)

$$f(a) + f(-a) = \frac{e^a}{1+e^a} + \frac{e^{-a}}{1+e^{-a}} = 1$$

$$I_1 = \int_{f(-a)}^{f(a)} xg[x(1-x)]\,dx$$

Apply king

$$I_1 = \int_{f(-a)}^{f(a)} (1-x)g\{(1-x)x\}\,dx$$

$$2I_1 = \int_{f(-a)}^{f(a)} g\{x(1-x)\}\,dx$$

$2I_1 = I_2$

$\frac{I_2}{I_1} = 2$

Q.12 (B)

Q.13 (C)

$$I = \int_a^1 x(1-x)^n dx$$

Alitar
Use king properly
Put $1 - x = t$
$dx = -dt$

$$= -\int_1^0 (1-x)t^n dt = \int_0^1 (1-x)t^n dt$$

$$= \int_0^1 (t^n - t^{n+1})dt = \left.\frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2}\right|_0^1 = \frac{1}{n+1} - \frac{1}{n+2}$$

Q.14 (C)
In first integral replace t by '1/t'

Q.15 (A)

$$I = \underbrace{\int_{-\pi/4}^{\pi/4} \underset{\text{odd fn.}}{\left\{\frac{e^x}{e^{2x}-1}\right\}} \underset{\text{Even Function}}{\sec^2 x}\,dx}_{\text{Odd function}}$$

I = 0

Q.16 (C)

$$f(x) = \begin{cases} e^{\cos x}\sin x & , -2 \le x \le 2 \\ 2 & , \text{otherwise} \end{cases}$$

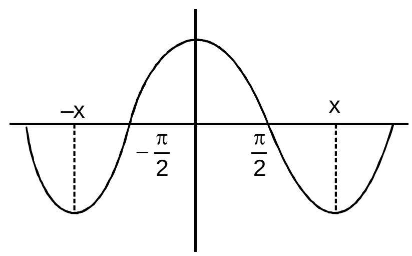
$$I = \int_{-2}^{3} f(x)\,dx + \int_{2}^{3} f(x)\,dx$$

$$= \int_{-2}^{2} e^{\cos x} \underset{\text{O as odd funtion}}{\sin x}\,dx + \int_2^3 2.dx = 2$$

Q.17 (C)

$$I = \int_{-2}^{-1}\left[x\left[1+\cos\left(\frac{\pi x}{2}\right)\right]+1\right]dx$$

$-2x < x < 1$



$-p < \frac{\pi x}{2} < \frac{\pi}{2}$

$-2 < x < -1$

$-\pi < \frac{\pi x}{2} < 0$

$-1 < \cos\frac{\pi x}{2} < 0$

$0 < 1 + \cos\frac{\pi x}{2} < 1$

$\left[1+\cos\frac{\pi x}{2}\right] = 0$

$$\int_{-2}^{-1} 1.dx + \int_{-1}^{0} [x+1]dx + \int_0^1 [x+1]dx$$

$-\frac{\pi}{2} < \frac{\pi x}{2} < 0$ $\qquad$ $0 < \frac{\pi x}{2} < \frac{\pi}{2}$

$1 < \cos\frac{\pi x}{2} + 1 < 2$ $\qquad$ $1 < \cos\frac{\pi x}{2} + 1 < 2$

$= 1 + 0 + 1 = 2$

Q.18 (B)

$$I = \int_0^2 x^3\left[1+\cos\frac{\pi x}{2}\right]dx$$

$$= \int_0^1 x^3\left[1+\cos\frac{\pi x}{2}\right]dx + \int_0^2 x^3\left[1+\cos\frac{\pi x}{2}\right]dx$$

$0 < x < 1$ $1 < x < 2$

$0 < \frac{\pi}{2}x < \frac{\pi}{2}$ $\frac{\pi}{2} < \frac{\pi}{2}\ x < \pi$

$1 > \cos\frac{\pi}{2}\ x > 0$ $0 > \cos\frac{\pi}{2}\ x > -1$

$1 < 1 + \cos\frac{\pi}{2}\ x > 0$ $0 < 1 + \cos\frac{\pi}{2}\ x < 1$

$[1 + \cos\frac{\pi}{2}x] = 1$ $[1 + \cos\frac{\pi}{2}x] = 0$

$$= \int_0^1 x^3dx + \int_1^2 x^3(0)dx = \frac{1}{4}$$

Q.19 (A)

$$\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^p.\frac{1}{n}$$

$$\int_0^1 x^pdx = \frac{x^{p+1}}{p+1}\Bigg|_0^1 = \frac{1}{p+1}$$

Q.20 (C)

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}(3\sqrt{r}+4\sqrt{n})^2}$$

$$\lim_{n\to\infty}\sum_{r=0}^{n}\frac{\sqrt{n}/\sqrt{n}}{\frac{\sqrt{r}}{\sqrt{n}}n\left(4+3\sqrt{\frac{r}{n}}\right)^2}$$

$$\int_0^1\frac{dx}{\sqrt{x}(4+3\sqrt{x})^2}$$

$r + 3\sqrt{x} = t$

$\frac{3}{2\sqrt{x}}\ dx = dt$

$$\frac{dx}{\sqrt{x}} = \frac{2}{3}dt = \frac{2}{3}\int\frac{dt}{t^2} = \frac{-2}{3}\ \frac{1}{t}$$

$$= -\frac{2}{3}\ \frac{1}{(4+3\sqrt{x})}\Bigg|_0^1 = \frac{-2}{3}\left[\frac{1}{n}-\frac{1}{4}\right]$$

$$= \frac{-2}{3}\left[\frac{4-7}{28}\right] = \frac{2}{28} = \frac{1}{14}$$

Q.21 (A)

$$\int_{\sin x}^{1} t^2f(t)dt = 1-\sin x$$

Applying Leibitz Rule

$-\sin^2x\ f(\sin t)\ \cos t = -\cos t$

$$f(\sec x) = \frac{1}{\sin^2 x}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2} = 3$$

$$\int_{\sin x}^{1} t^2 + \underset{\downarrow\atop 1}{ft}\,dt = 1 - \sin x$$

$$f(t) = \frac{1}{t^2} \Rightarrow f\left(\frac{1}{\sqrt{3}}\right) = 3$$

Q.22 (C)

In (0, 1)

$x^2 > x^3$

$$2^{x^2} > 2^{x^3}$$

$$\int_0^1 2^{x^2}\ dx > \int_0^1 2^{x^3}\ dx$$

$I_1 > I_2$

Q.23 (C)

$$U_n = \int_0^{n/2} x^n.\sin x dx$$

$$U_{10} = \int_0^{x/2} x^{10}.\sin x dx$$

$$= -x^{10}\ \cos x\,|_0^{x/2} + \int_0^{x/2} 10x^9\cos dx$$

$$= -\left(\frac{\pi}{2}\right)^{10} . \cos\frac{\pi}{2} + 10\int_0^{x/2} x^9 \cos x dx$$

$$= 10\ [x^9 \sin x|_0^{x/2} - 9\ 9\int_0^{x/2} x^8 \sin x dx$$

$$U_{10} = \left(\frac{\pi}{2}\right)^9 - 90\ U_8$$

$$U_{10} + 90\ U_8 = 10\left(\frac{\pi}{2}\right)^9$$

Q.24 (A)

$$I_n = \int_0^{\pi/4} \tan^2 x dx$$

$$\frac{1}{I_2 + I_4} = \frac{1}{\int_6^{\pi/4} (\tan^2 x + \tan^4 x) dx}$$

$$= \frac{1}{\int_6^{\pi/4} \tan^2 x + \sin^2 x dx} = 3$$

$$\frac{1}{I_3 + I_5} = 4$$

$$\frac{1}{I_4 + I_6} = 5 \qquad \text{A.P.}$$

Q.25 (A)

$$I = \int_1^2 f'(x) dx = f(z) - f(1) = 0$$

Q.26 (C)
Use by parts.

Q.27 (B)

$$f(x) = \int_0^x (t^2 - t + 1) dt \quad \forall\ x \in (3, 4)$$

$$\text{Geatest} = \int_0^4 (t^2 - t + 1) dt$$

$$\text{Least} = \int_0^3 (t^2 - t + 1) dt$$

$$\text{diff.} = \int_0^4 (t^2 - t + 1) dt - \int_0^3 (t^2 - t + 1) dt$$

$$= \int_3^4 (t^2 - t + 1) dt = \frac{t^3}{3} - \frac{t^3}{2} + t\Big|_3^4 = \frac{59}{6}$$

JEE-ADVANCED
MCQ/COMPREHENSION/COLUMN MATCHING

Q.1 (A, C)

$$\int_0^1 \left[\frac{x^2 + 2x + 2}{(x+1)(x^2 + 2x + 2)} + \frac{x^2 + x + 1}{(x+1)(x^2 + 2x + 2)}\right] dx$$

$$= \int_0^1 \frac{1}{x+1} dx + \int_0^1 \frac{x^2 + x + 1}{(x+1)(x^2 + 2x + 2)} dx$$

$$= \int_0^1 \frac{1}{x+1} dx + \int_0^1 \frac{dx}{x^2 + 2x + 2} + \int_0^1 \frac{x^2 + 2x + 2 - 2x}{(x^2 + 2x + 2)(x+1)}$$

$$= \int_0^1 \frac{2}{x+1} dx - \int_0^1 \frac{dx}{(x+1)^2 + 1}$$

$$= 2\left[\ln(x+1)\right]_0^1 - \left[\tan^{-1}(x+1)\right]_0^1$$

$$= 2\ell n2 - \tan^{-1}2 + \tan^{-1} 1 = \frac{\pi}{4} + 2\ell n2 - \tan^{-1} 2$$

$$= 2\ell n2 - [\tan^{-1} 2 - \tan^{-1} 1] = 2\ell n2 - \cot^{-1} 3$$

$$= \frac{\pi}{4} + 2\ell n2 - \frac{\pi}{2} + \cot^{-1} 2 = \frac{-\pi}{4} + 2\ell n2 + \cot^{-1} 2$$

Q.2 (A, B)

$$I = \frac{\pi}{2}\int_0^{\pi} f(\sin x) dx$$

Queen

$$I = \pi \int_0^{\pi/2} f(\sin x) dx$$

Q.3 (A, C)

$$I = \int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx \qquad ...(1)$$

$$\text{Put } x = \frac{1}{4} \Rightarrow dx = -\frac{dt}{t^2}$$

$$= -\int_{\infty}^{\infty}\frac{\frac{1}{t}}{\left(1+\frac{1}{t}\right)\left(1+\frac{1}{t^2}\right)}\cdot\frac{dt}{t^2} = \int_0^{\infty}\frac{dt}{(1+t)(1+t^2)}$$

$$I = \int_0^{\infty}\frac{dx}{(1+x)(1+x^2)} \qquad ...(2)$$

(1) + (2)

$$2I = \int_0^{\infty}\frac{dx}{(1+x^2)} = \tan^{-1}x\,|_0^{\infty}$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

Q.4 (A, B, C, D)

$$I = \int_a^b\frac{|x|}{x}dx \qquad \therefore \quad a < b$$

$x > 0$ $\qquad I = \int_a^b\frac{x}{x}dx = b - a$

$x < 0$ $\qquad I = -\int_a^b dx = a - b$

Q.5 (A, D)

$$f(x) = \int_b^x(\cos^4 t + \sin^{-1} t)\,dt$$

$$= f(x + x) = \int_b^{\pi+x}(\cos^4 t + \sin^{-1} t)\,dt$$

$$= f(x) + \int_b^x(\cos^4 t + \cos^{-1} t)\,dt$$

$$= f(x) + f(x)$$

$$= f(x) + \int_0^{\pi}(\cos^4 t + \cos^4 t)\,dt$$

$$= f(x) + \frac{1}{2}\int_0^{\pi/2}(\cos^4 t + \sin^4 t)\,dt$$

Q.6 (C, D)

$f(\sin^2 x) = \cos^2 x = 1 - \sin^2 x$

$\therefore \quad f'(t) = 1 - t$

$\Rightarrow \quad f(t) = t - \frac{t^2}{2} + c$

$1 = f(1) = 1 - \frac{1}{2} + c \Rightarrow c = \frac{1}{2}$

$\therefore \quad f(x) = 4 - \frac{x^3}{2} + \frac{c}{2}$

Q.7 (A, B)

$$f(x) = \int_0^x(2\cos^2 3t + 3\sin^2 3t)dt$$

$$= \int_0^x(2 + \sin^2 3t)dt$$

$$f(x + \pi) = \int_0^x(2 + \sin^2 3t)dt + \int_x^{x+\pi}(2 + \sin^2 3t)dt$$

$$= f(x) + \int_x^x(2 + \sin^2 3t)dt$$

$= f(x) + f(x)$

$$f(x) = \int_0^x(2 + \sin^2 3t)dt \qquad = 2\int_0^{\pi/2}(2 + \sin^2 3t)dt$$

$$= 2f\left(\frac{\pi}{2}\right)\ f(x + \pi)$$

$$= 2f\left(\frac{\pi}{2}\right) + f(x)$$

Q.8 (A, B, C)

$$I = \int_0^{2\pi}\sin^2 x\,dx \qquad \text{period is } \pi$$

$$= \pi\int_0^{\pi}\sin^2 x\,dx = 4\int_0^{\pi/2}\sin^2 x\,dx = 4\int_0^{\pi/2}\cos^2 x\,dx$$

Q.9 (A, B, C)

$f(-x) = -f(x) \qquad ... (1)$

$f(x + 2) = f(x) \qquad ... (2)$

$$g(2n) = \int_0^{2n}f(t)\,dt = n\int_0^2 f(t)\,dt$$

$\Rightarrow \quad g(2n) = n\,g(2) \qquad ... (3)$

$$\text{Now } g(-x) = \int_0^{-x}f(t)\,dt$$

put t = – z ⇒ dt = – dz

$= \int_0^x f(-z)(-dz) = -\int_0^x f(-z)\,dz$ (from (1))

$= \int_0^x f(t)\,dt = g(x)$

∴ g(– x) = g(x)

Again $g(x+2) = \int_0^{x+2} f(t)\,dt + \int_x^{x+2} f(t)\,dt$

∴ $g(x+2) = \int_0^x f(t)\,dt + \int_0^2 f(t)\,dt$ (∵ f → period)

⇒ g(x + 2) = g(x) + g(2) .... (4)

Putting x = 0, 2, ...

g(2) = g(0 + g(2) ⇒ g(0) = 0

g(4) = g(2) + g(2) ⇒ g(4) = 2g(2)

putting x → □ – x we get

g(2 – x) = g(– x) + g(2) = g(x) + g(2)

at x = 2

g(0) = 2g(2) ⇒ g(2) = 0

∴ g(0) = g(± 2) = g(± 4) = ... = 0

from (3)

g(2n) = 0

& from (4) g(x + 2) = g(x) ⇒ prd. of g(x) is 2

Q.10 (A, B, C, D)

$f(x) = f^{\{x\}}$

{x} is perido with = 1

(A)

(B) $\int_0^1 2^{\{x\}}dx = \int_0^1 2^x dx = \left.\frac{2^x}{\ell n2}\right|_0^1 = \frac{1}{\ell n2}$

(C) $\int_0^1 2^{\{x\}}dx = \frac{1}{\ell n2} = \frac{\ell n_e \rho}{\ell n e^2}$

(D) $\int_0^{100} 2^{\{x\}}dx = 100 \int_0^1 2^{\{x\}}dx = 100\ \ell n_2 \rho$

Q.11 (A, B)

$I_n = \int_0^1 \frac{dx}{(1+x^2)^n} = \int_0^1 (1+x^2)^{-n}dx$

$= \left[\frac{x}{(1+x^2)^n}\right]_0^1 - \int_0^1 (-n)(1+x^2)^{-n-1} 2x\ dx$

$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}$

$= \frac{1}{2^n} + 2n\int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}}\,dx$

$= \frac{1}{2^n} + 2n\ I_n - 2n\ I_{n+1}$

∴ $2n\ I_{n+1} = 2^{-n} + (2n - 1)\ I_n$

∴ $2I_2 = \frac{1}{2} + I_1 = \frac{1}{2} + \left[\tan^{-1}x\right]_0^1$

⇒ $I_2 = \frac{1}{4} + \frac{\pi}{8}$

Q.12 (B,C)

$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_1^2 f(x)\,dx$

$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r+n}{n}\right) = \int_0^1 f(1+x)\,dx$

$= \int_1^2 f(t)\,dt = \int_1^2 f(x)\,dx$

$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{2n} f\left(\frac{r}{n}\right) = \int_0^2 f(x)\,dx$

Comprehension # 1 (Q. No. 13 to 15)

Q.13 (C)

At x = 1, y = 0

$\frac{dy}{dx} = 2x \cdot x^8 - x^4 = 2 - 1 = 1$

∴ equation of tangent y – 0 = 1 (x – 1)

y = x – 1

Q.14 (A)

$F(x) = \int_1^x e^{t^2/2}\ (1 - t^2)\ dt$

$F'(x) = \left(e^{\frac{x^2}{2}}(1-x^2)\right)$

∴ $F'(x) = e^{1/2} \cdot 0 = 0$

Q.15 (A)

$\frac{dy}{dx} = 4x^3 (\ell n\ x^4)^2 - 3x^2 (\ell n\ x^3)^2$
$= 4x^3 (4\ \ell n\ x)^2 - 3x^2 (3\ \ell n\ x)^2$
$= 64x^3 (\ell n\ x)^2 - 27x^2 (\ell n\ x)^2$

$\therefore \lim_{x\to 0^+} \frac{dy}{dx} = 64 \lim_{x\to 0^+} x^3 (\ell n\ x)^2 - 27 \lim_{x\to 0^+} x^2 (\ell n\ x)^2 = 0$

Comprehension # 2 (Q. No. 16 to 18)

Q.16 (D)

$\because \frac{x^2+2}{x^2+1} = 1 + \frac{1}{x^2+1}$ = 1 + proper fraction

$\therefore \left[\frac{x^2+2}{x^2+1}\right] = 1$

Then, $\int_0^{10} \left[\frac{x^2+2}{x^2+1}\right] dx = \int_0^{10} 1.dx = 10$

Q.17 (C)

$\int_0^1 \sin([x]+[2x])\,dx =$

$\int_0^{1/2} \sin([x]+[2x])\,dx + \int_{1/2}^1 \sin([x]+[2x])\,dx$

$= 0 + \int_{1/2}^1 \sin(0+1)\,dx = \frac{\sin 1}{2}$

Q.18 (D)

$\int_{-1}^1 [|x|]\, d\left(\frac{1}{1+e^{-1/x}}\right)$

$= \int_{-1}^0 [-x]\, d\left(\frac{1}{1+e^{-1/x}}\right)$

$+ \int_0^1 [x]\, d\left(\frac{1}{1+e^{-1/x}}\right) = 0 + 0 = 0$

Comprehension # 3 (Q. No. 19 to 21)

Q.19 (D)

Q.20 (A)

Q.21 (C)

(19 to 21)

For the integral to be defined in $(0, \pi)$, $-1 < x < 1$

$f'(x) = \int_0^\pi \frac{1}{1+x\cos\theta}\,d\theta = \int_0^\pi \frac{1}{1-x\cos\theta}\,d\theta$

$\left(\because \int_0^\pi g(\theta)\,d\theta = \int_0^\pi g(\pi-\theta)\,d\theta\right)$

$\Rightarrow 2f'(x) = \int_0^\pi \frac{2}{1-x^2\cos^2\theta}\,d\theta$

$f'(x) = \int_0^\pi \frac{d\theta}{1-x^2\cos^2\theta} = 2\int_0^{\frac{\pi}{2}} \frac{d\theta}{1-x^2\cos^2\theta}$

$= 2\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\,d\theta}{1+\tan^2\theta-x^2} = 2\int_0^\infty \frac{dt}{t^2+1-x^2}$

$= 2\frac{1}{\sqrt{1-x^2}}\left[\tan^{-1}\frac{t}{\sqrt{1-x^2}}\right]_0^\infty = \frac{\pi}{\sqrt{1-x^2}}$

$\Rightarrow f(x) = \pi\sin^{-1} x + k$

but $f(0) = \int_0^\pi \frac{\ell n 1}{\cos\theta}\,d\theta = 0$ $\therefore f(x) = \pi\sin^{-1} x$

Range of $f(x) = \left(\frac{-\pi^2}{2}, \frac{\pi^2}{2}\right)$

$\left(\because \text{range of } \sin^{-1}x = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right)$

f(x) is differentiable in the interior of its domain and $f'(x) = 0$ has no solution.

Hence f(x) has no critical points.

$f(x) = \pi\sin^{-1} x$, $x \in (-1, 1)$

Applying Lagrange's theorem

$f'(x) = \frac{f(1)-f(-1)}{1-(-1)}$

$\Rightarrow \frac{\pi}{\sqrt{1-x^2}} = \frac{\pi^2}{2} \Rightarrow x = \pm\frac{\sqrt{\pi^2-4}}{\pi} \in (-1, 1)$

$\therefore$ There are two Lagrange's constant for f(x) in its domain.

Q.22 (A) → (r), (B) → (r), (C) → (p), (D) → (p)

(A) $5 < x < 10 \Rightarrow 0 < x - 5 < 5$

$\Rightarrow 0 < \frac{x-5}{5} < 1 \therefore \left[\frac{x-5}{5}\right] = 0$

(B) $-\tan 1 < x < 0 \Rightarrow \tan(-1) < x < 0$

$\Rightarrow -1 < \tan^{-1} x < 0 \Rightarrow 0 < -\tan^{-1} x < 1$

$\therefore [-\tan^{-1} x] = 0$

(C) $\frac{\pi}{6} < x < \frac{\pi}{3} \Rightarrow \frac{1}{2} < \sin x < \frac{\sqrt{3}}{2}$

$\Rightarrow 1 < 2 \sin x < \sqrt{3}$

$\therefore [2 \sin x] = 1$

(D) $I = \int_{-1}^{1} \frac{\cot^{-1} x}{\pi} dx \Rightarrow \int_{-1}^{1} \frac{\cot^{-1}(-x)}{\pi} dx$

$\Rightarrow \int_{-1}^{1} \frac{\pi - \cot^{-1} x}{\pi} dx$

$\Rightarrow I = \int_{-1}^{1} 1 dx - \int_{-1}^{1} \frac{\cot^{-1} x}{\pi} dx$

$I = x|_{-1}^{1} - I$

$2I = 1 - (-1) = 2$

Q.23 (A) → Q,S,T (B) → P,T (C) → R,S,Ts

$= \int_{\alpha}^{\pi/2-\alpha} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$ ....(i)

$I = \int_{\alpha}^{\pi/2-\alpha} \frac{\sin^n\left(\frac{\pi}{2}-\alpha\right)}{\sin^n\left(\frac{\pi}{2}-\alpha\right) + \cos^n\left(\frac{\pi}{2}-\alpha\right)} dx$

$= \int_{\alpha}^{\pi/2-\alpha} \frac{\cos^n \alpha}{\cos^n \alpha + \sin^n \alpha} dx$ .....(ii)

Adding Eqs. (i) and (ii), then

$2I = \int_{\alpha}^{\pi/2-\alpha} 1\, dx = \frac{\pi}{2} - 2\alpha$

$\therefore I = \frac{\pi}{4} - \alpha$ (Q, S, T)

(B) $I = \int_{-\pi}^{\pi} \frac{\sin^2 x}{1+\alpha^x} dx$ ......(i)

$I = \int_{-\pi}^{\pi} \frac{\sin^2(0-x)}{1+a^{0-x}} dx$

$= \int_{-\pi}^{\pi} \frac{\alpha^x \sin^2 x}{1+\alpha^x} dx$ .....(ii)

Adding Eqs. (i) and (ii), then

$2I = \int_{-\pi}^{\pi} \sin^2 x\, dx = 2\int_{0}^{\pi} \sin^2 x\, dx$

$\Rightarrow I = \int_{0}^{\pi} \sin^2 x\, dx = 2\int_{0}^{\pi/2} \sin^2 x\, dx$

$= 2. \frac{1}{2} . \frac{\pi}{2} = \frac{\pi}{2}$ (P, T)

(C) $I = \int_{\alpha}^{2\pi-\alpha} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$ ......(i)

$= \int_{\alpha}^{2\pi-\alpha} \frac{(2\pi - x)\sin^{2n}(2\pi - x)}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)} dx$

$= \int_{\alpha}^{2\pi-\alpha} \frac{(2p - x)\sin^{2n}}{\sin^{2n} x + \cos^{2n} x} dx$ ....(ii)

Adding Eqs. (i) and (ii), then

$2I = 2\pi \int_{\alpha}^{2\pi-\alpha} 1.dx = 2\pi(2\pi - \alpha - \alpha)$

$\therefore I = 2\pi^2 - 2\pi \quad \alpha$ (R, S, T)

Q.24 (A) → (q), (B) → (r), (C) → (p), (D) → (s)

(A) $\int_{0}^{\pi/2} \ell n (\tan x + \cot x)\ dx$

$= \int_{0}^{\pi/2} -\ell n (\sin x \cdot \cos x)\ dx$

$= -\int_{0}^{\pi/2} \ell n \sin x\, dx - \int_{0}^{\pi/2} \ell n \cos x\, dx$

$= -2\left(-\frac{\pi}{2}\ell n 2\right) = \pi\ \ell n\ 2$

(B) $I = \int_{0}^{\pi/2} \frac{\sin x - \cos x}{(\sin x + \cos x)^2} dx$

$= \int_{0}^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{\left(\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)\right)^2} dx$

$= \int_{0}^{\pi/2} \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx = -I$

$\therefore I = 0$

(C) $I = \int_{0}^{2\pi} x \sin^2 x \cos^2 x\, dx$

$= \int_{0}^{2\pi} (2\pi - x) \sin^2 x \cos^2 x\, dx$

$I = \pi \int_{0}^{2\pi} \sin^2 x \cos^2 x\, dx$

$= \frac{\pi}{4}\int_0^{2\pi} 4\sin^2 x\cos^2 x\,dx = \frac{\pi}{4}\int_0^{2\pi}\sin^2 2x\,dx$

$= \frac{\pi}{8}\int_0^{2\pi}(1-\cos 4x)\,dx = \frac{\pi}{8}\left(x - \frac{\sin 4x}{4}\right)\Big|_0^{2\pi}$

$= \frac{\pi}{8}(2\pi) = \frac{\pi^2}{4}$

(D) $\int_0^{\pi/2}(2\,\ell n\sin x - \ell n\,2 - \ell n\sin x - \ell n\cos x)$ dx

$= -\int_0^{\pi/2}\ell n\,2\,dx = -\frac{\pi}{2}\,\ell n\,2$

NUMERICAL VALUE BASED

Q.1 [61]

Note that $(x+1)^3 - (x-1)^3 = 2(3x^2+1)$

Hence $I = \frac{1}{2}\left[\int_2^4 \frac{(x+1)^3-(x-1)^3}{(x+1)^3(x-1)^3}dx\right]$

$= \frac{1}{2}\left[\int_2^4 \frac{dx}{(x-1)^3} - \int_2^4 \frac{dx}{(x+1)^3}\right]$

$= \frac{1}{2}\left[\frac{1}{2(x+1)^2} - \frac{1}{2(x-1)^2}\right]_2^4$

$= \left[\left(\frac{1}{25}-\frac{1}{9}\right)-\left(\frac{1}{9}-1\right)\right] = \frac{46}{225}$

Q.2 [64]

$U = \int_{\pi/6}^{\pi/2}\cos x\,dx = \sin x\Big|_{\pi/6}^{\pi/2} = \frac{1}{2}$

$V = \int_{-3}^{1}(-x^2)dx + \int_1^5 x^2 dx = \frac{1}{3}\left[-(1+27)+(125-1)\right]$

$= \frac{1}{3}[-28+124] = 32$

$\therefore\quad V = \lambda U \quad\Rightarrow\quad \lambda = 64$

Q.3 [10]

$\int_1^{100}\frac{f(x)}{x}dx = \int_1^{10}\frac{f(x)}{x}dx + \int_{10}^{100}\frac{f(x)}{x}dx = 5$

$+\int_{10}^{1}\frac{f\left(\frac{100}{t}\right)}{100}.t\left(\frac{-100dt}{t^2}\right)$ (substituting $x = \frac{100}{t}$)

$= 5 + \int_t^{10} f\left(\frac{100}{t}\right).\frac{dt}{t} = 5 + \int_1^{10}\frac{f(t)}{t}dt = 10$

Q.4 [29]

$$\frac{2005\int_0^{1002}\frac{\sqrt{1003^2-x^2}-\sqrt{1002^2-x^2}}{2005} + \int_{1002}^{1003}\sqrt{1003^2-x^2}\,dx}{\int_0^1\sqrt{1-x^2}\,dx}$$

$$= \frac{\int_0^{1003}\sqrt{1003^2-x^2}\,dx - \int_0^{1002}\sqrt{1002^2-x^2}\,dx}{\int_0^1\sqrt{1-x^2}\,dx}$$

$$= \frac{(1003^2-1002^2)\left(\frac{\pi}{4}\right)}{\left(\frac{\pi}{4}\right)} = 2005$$

Hence $2^2 + 5^2 = 29$

Q.5 [4]

$I = \int_0^{\pi/2}\sqrt{\sin 2\theta}\,\cos\theta d\theta$; $I = \int_0^{\pi/2}\sqrt{\sin 2\theta}\,\sin\theta d\theta$

(By property)

Let $\sin\theta - \cos\theta = t$; 2I

$= \int_{-1}^{1}\sqrt{1-t^2} = 2\int_0^1\sqrt{1-t^2}\,dt$; $I = \int_0^1\sqrt{1-t^2}\,dt = \frac{\pi}{4}$

Q.6 [4]

Substitute $x = \tan\theta$ in I_2.

$I_2 = \int_0^{\pi/4}\frac{d\theta}{(1+\tan\theta)^2} = \int_0^{\pi/4}\frac{d\theta}{\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)^2}$

$\left[\int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

$= \int_0^{\pi/4}\frac{(1+\tan\theta)^2}{4}d\theta \qquad \Rightarrow I_2 = \frac{I_1}{4}$

Q.7 [2]

$I = \int_0^1 \underbrace{e^{t^2+t}}_{f(t)} + \underbrace{t.e^{t^2+t}(2t+1)}_{f'(t)}dt = t.e^{t^2+t}\Big|_0^1 = e^2$

Q.8 [2]

Let $I = \int_{-1}^{0} \frac{xe^{-x}}{(x+1)e^{-x}+1}dx$ substituting $1 + (x + 1)e^{-x}$

$= t \Rightarrow -xe^{-x}\,dx = dt$

$I = -\int_{1}^{2} \frac{dt}{t} = -\ell n 2$

Q.9 [0]

$I = \int_0^a f(x)\,g(x)\,h(x)\,dx$

$I = \int_0^a f(a-x)\,g(a-x)\,h(a-x)\,dx$

$= \int_0^a f(x)\,.\,(-g(x))\,\frac{3h(x)-5}{4}\,dx$

$I = -\frac{3}{4}\int_0^a f(x)\,g(x)\,h(x) + \frac{5}{4}\int_0^a f(x)\,g(x)\,dx$

$I = -\frac{3}{4}\,I + 0 \quad \frac{7}{4}I = 0$

$\frac{7}{4}\,I = 0 \qquad \Rightarrow \qquad I = 0$

$\because \qquad f(x)\,g(x)\,dx = 0$

$I_1 = \int_0^a f(x)\,g(x)\,dx$ ......(i)

$I_1 = \int_0^a f(a-x)\,g(a-x)\,dx$

$I_1 = \int_0^a f(x)\,(-g(x))\,dx$ ......(ii)

(i) + (ii) $\quad 2I_1 = 0 \;\Rightarrow\; I_1 = 0$

Q.10 [2]

$I = \int_0^{\pi} f(x)\,dx = \int_0^{\pi} \frac{\sin x}{x}\,dx$ ... (i)

$I = \int_0^{\pi} f(\pi-x)\,dx = \int_0^{\pi} \frac{\sin(\pi-x)}{\pi-x}dx$

$= \int_0^{\pi} \frac{\sin x}{\pi-x}\,dx$ ... (ii)

(i) + (ii)

$\Rightarrow \quad 2I = \int_0^{\pi}\left\{\frac{\sin x}{x} + \frac{\sin x}{\pi-x}\right\}dx$

$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{x(\pi-x)}dx$ ... (iii)

Now $\quad \frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(x)\,f\left(\frac{\pi}{2}-x\right)dx$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,x\,\frac{\sin\left(\frac{\pi}{2}-x\right)}{\frac{\pi}{2}-x}dx$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\cdot\frac{\cos x}{\frac{\pi}{2}-x}\,dx = \frac{\pi}{4}\cdot\int_0^{\pi/2} \frac{\sin 2x}{x\left(\frac{\pi}{2}-x\right)}\,dx$

$= \frac{\pi}{8}\int_0^{\pi} \frac{\sin t}{\frac{t}{2}\left(\frac{\pi}{2}-\frac{t}{2}\right)}dt$, where $t = 2x$ ... (iv)

(iii) + (iv)

$\Rightarrow \quad \frac{\pi}{2}\int_0^{\pi} f(x)\,f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi} f(x)\,dx$

Q.11 [8]

Let

$I = \frac{6}{a^2+b^2}\int_0^{\pi/2} \frac{(a^2+b^2)(a^2\sin^2 x + b^2\cos^2 x)}{a^4\sin^2 x + b^4\cos^2 x}dx$

$= \frac{6}{3\pi}.4\int_0^{\pi/2}\left(\frac{a^4\cos^2 x + b^4\cos^2 x + a^2b^2}{a^4\sin^2 x + b^4\cos^2 x}\right)dx$

$= \frac{8}{\pi}\int_0^{\pi/2} dx + \frac{8a^2b^2}{\pi}\int_0^{\pi/2} \frac{\sec^2 x}{a^4\sin^2 x + b^4\cos^2 \pi}dx$

$= 4 + 8\frac{b^2}{a^2}$

$\int_0^{\pi/2} \frac{\sec^2 x\,dx}{\tan^2 x + (b^2/a^2)} = 4 + \frac{8b^2}{\pi a^2}\cdot\frac{a^2}{b^2}\cdot\tan^{-1}\left(\frac{a^2\tan^{-1}x}{b^2}\right)_0^{\pi/2}$

$= 4 + 4 = 8$

Q.12 [16]

$I = \int_0^{2\pi} |\sin(x+\alpha)|\,dx$, where $\tan\alpha = \frac{1}{\sqrt{15}}$

$= 4\int_\alpha^{2\pi+\alpha} |\sin|\,dt$ (substituting $x + \alpha = t$)

=

$\left(\int_a^{a+T} f(x)dx = \int_0^T f(x)dx \text{ if } f(x+T) = f(x)\right) = 16$

Q.13 [25]

Let $I = \int_{-\pi+a}^{3\pi+a} |x-a-\pi| \sin\left(\frac{x}{2}\right)dx$

Let $x - a - \pi = t$

$\Rightarrow I = \int_{-2\pi}^{2\pi} |t| \sin\left(\frac{t+a}{2} + \frac{\pi}{2}\right)dt$

$= \int_{-2\pi}^{2\pi} |t| \cos\left(\frac{a+t}{2}\right)dt$

$= \int_{-2\pi}^{2\pi} t\left\{\cos\left(\frac{a+t}{2}\right) + \cos\left(\frac{a-t}{2}\right)\right\}dt$

$= \int_{-2\pi}^{2\pi} t\left\{2\cos\frac{a}{2}\cos\frac{t}{2}\right\}dt = 2\cos\left(\frac{a}{2}\right)\int_0^{2\pi} \cos\left(\frac{t}{2}\right)dt$

Let $\frac{t}{2} = y$ $\quad 8\cos\left(\frac{a}{2}\right)\int_0^{\pi} y\cos y\,dy$

$= 8\cos\left(\frac{a}{2}\right)\{y\sin y + \cos y\}_0^{\pi} = -16\cos\left(\frac{a}{2}\right)$

Now $I = -16 \Rightarrow \cos\left(\frac{a}{2}\right) = 1 \Rightarrow a = 4k\pi, k \in I$

Hence number of value of 'a' is 25.

Q.14 [4]

$\int_0^1 \frac{dx}{1+x^2} = \int_0^1 (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots\ldots\ldots)\,dx$

$\Rightarrow \quad \tan^{-1}x\Big|_0^1 = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \ldots\ldots \Big|_0^1$

$\Rightarrow \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots\ldots$

Q.15 [65]

$f(x) = x + \int_0^1 t(x+t)\, f(t)dt$; f(x)

$= x + x\int_0^1 t\,f(t)\,dt + \int_0^1 t^2 f(t)\,dt$

Let $\int_0^1 t\,f(t)\,dt = a$ & $\int_0^1 t^2 f(t)\,dt = b$

Hence $f(x) = (a+1)x + b$

so $a = \int_0^1 t\{(a+1)t + b\}dt = \frac{a+1}{3} + \frac{b}{2}$

$\Rightarrow 4a - 3b = 2$ ....(1)

& $b = \int_0^1 t^2\{(a+1)t + b\}\,dt = \frac{a+1}{4} + \frac{b}{3}$

$\Rightarrow 8b - 3a = 3$ ....(2)

from (1) & (2) $a = \frac{25}{23}$ & $b = \frac{18}{23}$

so $\int_0^1 f(t)\,dt = \frac{6}{23}\int_0^1 (8t+3)\,dt = \frac{6}{23}.7 = \frac{42}{23}$

$\Rightarrow p + q = 65$

KVPY
PREVIOUS YEAR'S

Q.1 (B)

$\int_2^8 f(x)\,dx = 2 + 3 + 2 + 5 + 3 + 7 = 22$

Q.2 (B)

$I = 2012\int_0^1 \frac{e^{\cos\pi x}}{e^{\cos\pi x} + e^{-\cos\pi x}}dx$

using king property $I = 2012\int_0^1 \frac{e^{-\cos\pi x}}{e^{-\cos\pi x} + e^{\cos\pi x}}dx$

$\Rightarrow 2I = 2012 \Rightarrow I = 1006$

Q.3 (D)

$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2 - r^2}} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{4 - (r/n)^2}}$

$= \int_0^1 \frac{dx}{\sqrt{4 - x^2}} = \left(\sin^{-1}\left(\frac{x}{2}\right)\right)_0^1 = \frac{\pi}{6}$

Q.4 (B)

$$f'(x) = 3\left(x^2 - \frac{2}{3}x - \frac{1}{3}\right) = 3x^2 - 2x - 1$$

$f(x) = x^3 - x^2 - x - \lambda$

$f(B) = 8 - 4 - 2 + \lambda = 0 \Rightarrow \lambda = -2$

$f(x) = x^3 - x^2 - x - 2$

$$\int_{-1}^{1} f(x)dx = -2\int_0^1 (x^2+2)dx = -2\left(\frac{1}{3}+2\right) = \frac{-14}{3}$$

Q.5 (A)

$$In = \int_0^{1/2} \frac{x^n}{n!}dx + \int_{1/2}^{1} \frac{(1-x)^n}{n!}dx$$

$$= \frac{1}{(n+1)!}\left(\left(\frac{1}{2}\right)^{n+1} + \left(\frac{1}{2}\right)^{n+1}\right) = \frac{\left(\frac{1}{2}\right)^n}{(n+1)!}$$

$$\sum_{n=1}^{\infty} I_n = \left(\frac{1/2}{2!} + \frac{(1/2)^2}{3!} + ...\right) = 2\sqrt{e} - 3$$

Q.6 (D)

$$I_1 = \int_0^{\pi/2} (\sin x)^{\sqrt{2}}.\sin x\,dx$$

$$I_2 = \int_0^{\pi/2} (\sin x)^{\sqrt{2}-1}dx$$

$$I_1 = \left((\sin x)^{\sqrt{2}} \int \sin x\,dx\right)_0^{\pi/2} -$$

$$\int_0^{\pi/2} \left(\sqrt{2}(\sin x)^{\sqrt{2}-1}\cos x \int \sin x\,dx\right)$$

$$= -\left(\cos x(\sin x)^{\sqrt{2}}\right)_0^{\pi/2} + \sqrt{2}\int_0^{\pi/2} (\sin x)^{\sqrt{2}-1}(1-\sin^2 x)dx$$

$$\frac{I_1}{I_2} = \frac{\sqrt{2}}{1+\sqrt{2}} \times \frac{(\sqrt{2}-1)}{(\sqrt{2}-1)} = 2 - \sqrt{2}$$

Q.7 (D)

$$\int_{-2012}^{2012} (\sin(x)^3 + x^5 + 1)dx$$

$$= \int_{-2012}^{2012} (\sin(x^3)dx + \int_{-2012}^{2012} x^5dx + \int_{-2012}^{2012} dx = 4024$$

Q.8 (B)

$$\int_0^5 [x]\{x\}dx = \int_0^5 [x](x-[x])dx = \int_0^1 0.dx + \int_1^2 1.(x-1)dx +$$

$$\int_2^3 2(x-2)dx + \int_3^4 3(x-3)dx + \int_4^5 4(x-4)dx$$

$$= \left(\frac{(x-1)^2}{2}\right)_1^2 + 2\left(\frac{(x-2)^2}{2}\right)_2^3 + 3\left(\frac{(x-3)^2}{2}\right)_3^4 + 4\left(\frac{(x-4)^2}{2}\right)_4^5$$

$$= \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2} = 5$$

Q.9 (C)

$$I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}dx.................(1)$$

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

$$I = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}}dx$$

$$I = \int_{-\pi}^{\pi} \frac{a^x\cos^2 x}{1+a^x}dx......................(2)$$

add equation (A) and (B)

$$2I = \int_{-\pi}^{\pi} \cos^2 x\left(\frac{1}{1+a^x} + \frac{a^x}{1+a^x}\right)dx$$

$$I = \int_0^{\pi} \cos^2 x\,dx = 2\int_0^{\pi/2} \cos^2 x\,dx$$

$= \pi / 2$

Q.10 (B)

$L = \sqrt[3]{2012} + \sqrt[3]{2013} + ... + \sqrt[3]{3011}$ ....(A)

$R = \sqrt[3]{2013} + \sqrt[3]{2014} + ... + \sqrt[3]{3012}$ ....(B)

$$I = \int_{2012}^{3012} x^{1/3}dx \qquad \text{Let } f(x) = x^{1/3}$$

$$n = \frac{b-a}{h} = \frac{3012-2012}{1} = 1000$$

$$I = \frac{(b-a)}{n}[f(a) + f(a+h) + f(a+2h) + ...+ f(a+(b-1)h]$$

$=[f(2010) + f(2013) +.....+f(3011)]$

$I = (2012)^{1/3} + (2013)^{1/3} +.........+ (3011)^{1/3}$

$2I = 2(2012)^{1/3} + 2(2013)^{1/3} + ...+)2(3011)^{1/3}$

$= (2012)^{1/3} + (2012)^{1/3} + 2(2013)^{1/3} + ...+$ (B)

$(3011)^{1/3} + (3012)^{1/3} - (3012)^{1/3}$
$= (2012)^{1/3} + L + R - (3012)^{1/3}$
$2I < L + R$

Q.11 (A)
Let r be an integer in (–10, 10)

Now, LHL $= \lim_{x\to r^-} \int_{-10}^{x} 2^{[t]}dt$

$= \lim_{h\to 0^+}\left[\int_{-10}^{-9} 2^{[t]}dt + \int_{-9}^{-8} 2^{[t]}dt + ... + \int_{r-1}^{r-h} 2^{[t]}dt\right]$

$= \lim_{h\to 0}\left[2^{-10} + 2^{-9} + ... + 2^{r-1}(1-h)\right]$

$= 2^{-10} + 2^{-9} + ... + 2^{r-1}$ ...(A)

$\lim_{x\to r^+} \int_{-10}^{x} 2^{[t]}dt$

$= \lim_{h\to 0^+}\left[\int_{-10^2}^{-9} 2^{[t]}dt + \int_{-9}^{-8} 2^{[t]}dt + ... + \int_{r}^{r+h} 2^{[t]}dt\right]$

$= 2^{-10} + 2^{-9} + ...+ 2^{r-1}$ ...(B)

$f(r) = \int_{-10}^{r} 2^{[t]}dt$

$= 2^{-10} + 2^{-9} + ...+ 2^{r-1}$ ....(C)

From (A), (B) and (C)
f(x) is continous at all integers.

Q.12 (C)

$$\int_1^{n+1} \frac{\{x\}^{[x]}}{[x]}dx = \int_1^2 \frac{\{x\}^{[x]}}{[x]}dx + \int_2^3 \frac{\{x\}^{[x]}}{[x]}dx + ... + \int_1^{n+1} \frac{\{x\}^{[x]}}{[x]}dx$$

$$= \sum_{r=1}^{n} \int_r^{r+1} \frac{\{x\}^{[x]}}{[x]}dx$$

$$= \sum_{r=1}^{n} \int_r^{r+1} \frac{(x-r)^r}{r}dx$$

$$= \sum_{r=1}^{n} \left[\frac{(x-r)^{r+1}}{r(r+1)}\right]_r^{r+1}$$

$$= \sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

Q.13 (D)

$$\int_1^n [r]\{r\}dx = \sum_{r=1}^{n-1} \int_r^{r+1} r(x-r)dx$$

$$\sum_{r=1}^{n-1} r\left[\frac{x^2}{2} - rx\right]_r^{r+1}$$

$$\sum_{r=1}^{n-1} r\left[\frac{(r+1)^2 - r^2}{2} - r.1\right]$$

$$\sum_{r=1}^{n-1} r\left[\frac{1}{2}\right] = \frac{1}{2}\frac{n(n-1)}{2}$$

$$\frac{n(n-1)}{4} \geq 2013$$

$n(n-1) \geq 4 \times 2013$

$$\left(n - \frac{1}{2}\right)^2 \geq \frac{2013\times 16 + 1}{4}$$

$$n \geq \frac{\sqrt{32209}}{2} + \frac{1}{2}$$

least n = 91

Q.14 (B)

$$A(t) = \int_0^1 \{(\sin t)x^2 - (2\cos t)x + \sin t\}dx$$

$$A(t) = \frac{\sin t}{3} - \cos t + \sin t = \frac{4}{3}\sin t - \cos t$$

$$A'(t) = \frac{4\cos t}{3} + \sin t$$

St. I and IV are false
Ans. (B)

Q.15 (D)

$\because f(x) \geq 0$

$\int_0^1 f(x)^2 dx \leq 100$ not necessarily true.

because $(f(x))^2$ can take very high values then area bounded by $(f(x))^2$, x-axis & x = 0 to 1 may cross 100.

Q.16 (C)

$$f(x) = x + \int_0^x f(t)dt$$

$f'(x) = 1 + f(x) \Rightarrow f'(x) - f(x) = 1$

$\Rightarrow e^{-x}f'(x) - f(x)e^{-x} = e^{-x}$

$$\Rightarrow \frac{d}{dx}(f(x)e^{-x}) = e^{-x}$$

$$\Rightarrow f(x)e^{-x} = \frac{e^{-x}}{-1} + c$$

$\Rightarrow f(x) = -1 + ce^{x}$

$f(0) = 0 = -1 + ce^{o} \Rightarrow c = 1$

$f(x) = e^x - 1$

$f(x) + f(y) + f(x)f(y) = e^x - 1 + e^y - 1 + (e^x - 1)(e^y - 1)$

$= e^x - 1 + e^y - 1 + e^x.e^y - e^y - e^y - e^x + 1$

$= e^x.e^y - 1 = e^{x+y} - 1$

$= f(x+y)$

Q.17 (A)

$$I_n = \int_0^{\pi/2} \underset{I}{x^n} \underset{II}{\cos} \; x\,dx$$

$$= x_n \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1} \sin x\,dx$$

$$= \left(\frac{\pi}{2}\right)^n - O - (nx^{n-1}(-\cos x)\Big|_0^{\pi/2} - \int_0^{\pi/2} n(n-1)x^{n-2}(-\cos x)dx$$

$$= \left(\frac{\pi}{2}\right)^n - 0 - n(n-1)\int_0^{\pi/2} x^{n-2}\cos x\; dx$$

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

$$\sum_{n=2}^{\infty}\left(\frac{I_n}{n!} + \frac{I_{n-2}}{(n-2)!}\right) = \sum_{n=2}^{\infty}\left(\frac{\left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}}{n!} + \frac{I_{n-2}}{(n-2)!}\right)$$

$$= \sum_{n=2}^{\infty}\left(\left(\frac{\pi}{2}\right)^n \frac{1}{n!}\right) \quad = \left(\frac{\pi}{2}\right)^2 \frac{1}{2!} + \left(\frac{\pi}{2}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{2}\right)^4 \frac{1}{4!} + ...$$

$$= e^{\pi/2} - 1 - \left(\frac{\pi}{2}\right)$$

Q.18 (B)

$$\text{Let } I = \int_1^n [x]\left[\sqrt{x}\right]dx$$

$I \le x \le 4 \qquad \left[\sqrt{x}\right] = 1$

$4 \le x < 9 \qquad \left[\sqrt{x}\right] = 2$

$9 \le x < 16 \qquad \left[\sqrt{x}\right] = 3$

$$I = \int_1^2 dx + \int_2^3 2dx + \int_3^4 3dx + \int_4^5 8dx + \int_5^6 10dx + \int_6^7 12dx + \int_7^8 14dx + \int_8^9 16dx + \int_9^{10} 27dx + \int_{10}^{11} 30dx.....$$

$I = 1 + 2 + 3 + 8 + 10 + 12 + 14 + 16 = 66$

So $n = 9$

Q.19 (B)

$$\int_0^n \cos(2\pi[x]\{x\})\,dx$$

$$\int_0^1 \cos(0)dx + \int_1^2 \cos(2\pi(x-1))dx + \int_2^3 \cos(4\pi(x-2))dx$$

$$+........+ \int_{n-1}^{n} \cos(2\pi(n-1)(x-(n-1)))\,dx$$

$$= (1-0) + \int_1^2 \cos\, 2\pi x\,dx + \int_2^2 \cos\, 4\pi x\,dx$$

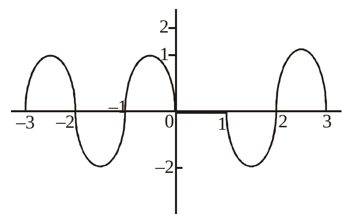
$$+......+ \int_{n-1}^{n} \cos\,(2\pi(n-1)\,\pi)x\,dx$$

$$= 1 + \frac{\sin 2\pi x}{2\pi}\Big|_1^2 + \frac{\sin 4\pi x}{4\pi}\Big|_2^3 + ...+ \frac{\sin 2\pi(n-1)x}{2\pi(n-1)}\Big|_{n-1}^{n}$$

$= 1 + 0 = 1$

Q.20 (D)

$f(x) = [x]\sin \pi x$



Not diff for $\forall\; x \in R$.

Not sym about x = 0.

$$\int_{-3}^{3} f(x)dx \ne 0$$

$f(x) = \alpha$ will have ∞ sol[n]

Q.21 (D)

$f(x) = \max\{|x|, |x-1|, ... ,|x-2n|\}$

$x \ge n$: $f(x)=|x|$ $\qquad$ $x < n$: $|x - 2n|$

$$\int_0^{2n} f(x)dx = \int_0^n f(x)dx + \int_n^{2n} f(x)dx$$

$$= \int_0^n |x - 2n|\,dx + \int_n^{2n} |x|\,dx$$

$$= \int_0^n (2n - x)\,dx + \int_n^{2n} x.dx$$

$$= \left[2nx - \frac{x^2}{2}\right]_0^n + \left[\frac{x^2}{2}\right]_n^{2n}$$

$$= \left(2n^2 - \frac{n^2}{2}\right) + \left(\frac{4n^2}{2} - \frac{n^2}{2}\right)$$

$$= \frac{3n^2}{2} + \frac{3n^2}{2} = 3n^2$$

Q.22 (D)

Let $P(x) = ax^3 + bx^2 + cx + d$

$a + b + c + d = 3$

$d = 2$

$-a + b - c + d = 4$

$2b + 2d = 7$

$2b + 4 = 7$

$2b = 3$

$$b = \frac{3}{2}$$

$$\int_{-1}^{1}\left(ax^3 + bx^2 + cx + d\right)dx$$

$$2\int_{0}^{1}\left(bx^2 + d\right)dx$$

$$= 2\left[b\frac{x^3}{3} + dx\right]_0^1$$

$$= 2\left(\frac{b}{3} + d\right)$$

$$= 2\left(\frac{1}{2} + 2\right) = 5$$

Q.23 (A)

$$f(x) = \int_0^\infty e^{-t}|x-t|dt$$

$$f(x) = \int_0^x e^{-t}|x-t|dt + \int_x^\infty e^{-t}(t-x)dt$$

$$f'(x) = e^{-x}(x-x) - e^{-0}(x-0).0 + \int_0^x e^{-t}(1)dt$$

$$+ 0 - e^{-x}(x-x).1 + \int_x^\infty -e^{-t}\,dt$$

$$= \left[-e^{-t}\right]_0^x + \left[e^{-t}\right]_x^\infty = -e^{-x} + 1 + 0 - e^{-x}$$

$$f'(x) = 1 - 2e^{-x}$$

$dy = 1 - 2e^{-x}dx$

$y = x + 2e^{-x} + c$

$f(x) = x + 2e^{-x} + c$

$$f(0) = \int_0^\infty e^{-t}t\,dt$$

$$= \left[\left(-e^{-t}\right)t\right]_0^\infty + \int_0^\infty e^{-t}dt$$

$$= \left[0 - e^{-t}\right]_0^\infty$$

$= 0 + 1$

$f(0) = 1$

$f(0) = 1 = 0 + 2e^{-0} + c$

$c = -1$

$f(x) = x + 2e^{-x} - 1$

Q.24 (B)

$$f(x) + \int_0^x t f(t)dt + x^2 = 0$$

$$f'(x) + xf(x) + 2x = 0$$

$$\frac{f'(x)}{f(x)+2} = -x$$

$$\int\frac{f'(x)}{(f(x)+2)}dx = -\int x dx$$

$$\ell n(f(x)+2) = -\frac{x^2}{2} + c$$

$$f(x) + 2 = e^{-x^2/2} + c$$

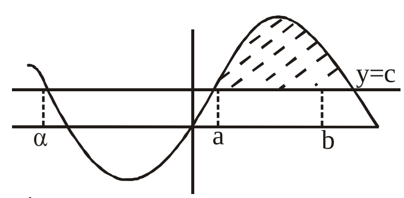
$$f(x) = k\,e^{-x^2/2} - 2$$

where $x = 0, f(x) = 0,\ k = 2$

$$f(x) = 2(e^{-x^2/2} - 1)$$

clearly $\lim\limits_{x\to\infty} f(x) = -2$

Q.25 (A)



$$\int_a^b (2x - 4x^3)dx = 2(b-a)c$$

$$(x^2 - x^4)_a^b = 2(b-a)c$$

$(a + b)(1 - (a^2 + b^2)) = 2c$

$(a + b)(1 - (a + b)^2 + 2ab) = 2c$ .....(A)

again $2x - 4x^3 = c$

$4x^3 - 2x + c = 0$ (roots a, b, α)

$a + b + \alpha = 0$ clearly $a + b = \alpha$ ....(B)

$ab + (a + b)\ \alpha = -\frac{1}{2}$ $ab = \alpha^2 - \frac{1}{2}$ ....(C)

$ab\alpha = -\frac{c}{4}$ $c = -4\alpha\left(\alpha^2 - \frac{1}{2}\right)$ ....(D)

put value of $(a + b)$, ab, c from eq. (B), (C), (D) in equation (A) and solve it

$$\left\{1-\alpha^2+2\left(\alpha^2-\frac{1}{2}\right)\right\}=-8\alpha\left(\alpha^2-\frac{1}{2}\right)$$

$1-\alpha^2+2\alpha^2-1=8\alpha^2-4$

$\alpha^2 = 8\alpha^2-4$

$7\alpha^2 = 4$

$$\alpha=\frac{2}{\sqrt{7}}$$

Q.26 (A)

$\therefore$ f(x) is always positive for $x \in [0, 1]$

$$\therefore f(x)x^2 \Rightarrow \int_0^1 f(x)dx < \int_0^1 x^2$$

$$\boxed{I<\frac{1}{3}}$$

But it is given $I=\frac{1}{3}$ which is not possible

Q.27 (D)

By Cauchy Schwarz inequality

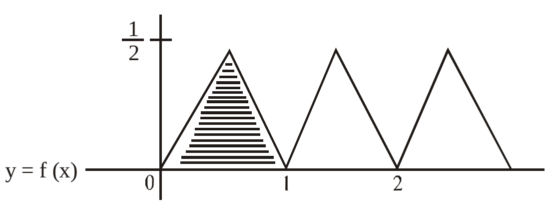
$$\left\{\int_a^b f(x)g(x)dx\right\}^2 \le \int_a^b (f(x))^2\,dx\int_a^b (g(x))^2\,dx$$

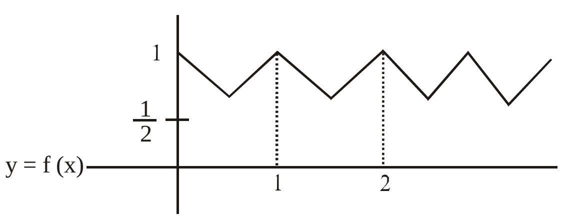
Here g (x) = 1

and equality holds only when $\frac{f(x)}{g(x)}=\lambda$

So, f (x) is constant

Q.28 (C)





$$\int_0^n g(x)dx = n\int_0^1 g(x)dx=\frac{3}{4}n\int_0^n f(x)dx=\frac{n}{4}$$

Q.29 (C)

Given integral can be distributed into

$$\int_{1/2}^{1/\sqrt{3}}\frac{1}{x^2}.dx+\int_{1/\sqrt{3}}^{\sqrt{3}}3dx+\int_{\sqrt{3}}^{2}x^2.dx=\frac{14}{3}$$

Q.30 (C)

$$I=\int_0^{\pi}1-|\sin 8x|dx$$

$$=\int_0^{\pi}(x|\sin 8x|)dx$$

$$=\pi-\int_0^{8.\frac{8}{\pi}}(x|\sin 8x|)dx$$

$$=\pi-\int_0^{\frac{8}{\pi}}x\sin 8x\,dx$$

$$=\pi-8\left[\frac{-\cos 8x}{8}\right]^{\frac{\pi}{8}}$$

$P+(-1-\phi)=\pi-2$

Q.31 (A)

$x^8 < x^4$ as $x \in (0, 1)$

$$\Rightarrow \frac{x}{1+x^8}>\frac{x}{1+x^4} \qquad \Rightarrow J>\int_0^1\frac{x\,dx}{1+x^4}$$

$$\Rightarrow J>\frac{1}{2}\int_0^1\frac{2x\,dx}{1+x^4}=\frac{1}{2}\int_0^1\frac{dt}{1+t^2}=\frac{\tan^{-1}1}{2}=\frac{\pi}{8} \Rightarrow J>\frac{\pi}{8}$$

Also $x^8<1 \Rightarrow 1+x^8<2 \Rightarrow \frac{x}{1+x^8}>\frac{x}{2}$

$$\Rightarrow J>\int_0^1\frac{x}{2}dx=\frac{1}{4}$$

Q.32 (A)

$p'(x) = g'(x) = \frac{2}{\pi} = f(x) - \frac{2}{\pi} = |\sin x| - \frac{2}{\pi}$

$\Rightarrow$ p(x) is not one-one

$p(x+\pi) = g(x+\pi) - \frac{2}{\pi}(x+\pi)$

$$=\int_0^{x+\pi}|\sin t|\,dt-\frac{2x}{\pi}-2$$

$$=\int_0^{x}|\sin t|\,dt-\frac{2x}{\pi}=g(x)-\frac{2x}{\pi}=p(x)$$

Q.33 (A)

$$f(x) = 2\int_0^x tf(t)\,dt + 1$$

$$\Rightarrow f'(x) = 2x f(x) \qquad \Rightarrow \frac{f'(x)}{f(x)} = 2x$$

$$\Rightarrow \ell nf(x) = x^2 + c \qquad \Rightarrow f(x) = Ke^{x^2}$$

$$f(0) = 1 \Rightarrow f(x) = e^{x^2} \qquad \Rightarrow f(1) = e$$

Q.34 (B)

$$I = \int_0^{\pi/2}\left(\frac{\sin^2 x}{1+e^x} + \frac{\sin^2 x}{1+e^{-x}}\right)dx$$

$$= \int_0^{\pi/2}\sin^2 x\,dx = \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}$$

Q.35 (C)

Since, $\sum_{k=0}^{n}\frac{1}{\sqrt{n^2+n}} \le \sum_{k=0}^{n}\frac{1}{\sqrt{n^2+k}} \le \sum_{k=0}^{n}\frac{1}{\sqrt{n^2+0}}$

$$\Rightarrow \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}} \le \lim_{n\to\infty} S_n \le \lim_{n\to\infty}\frac{n}{\sqrt{n^2}}$$

$$\Rightarrow 1 \le \lim_{n\to\infty} S_n \le 1$$

$$\Rightarrow \lim_{n\to\infty} S_n \le 1$$

Q.36 (B)

$f(x)$ is an increasing function.

so, $f(x) \in \left[1, \frac{1}{\sin 1}\right) \ \forall x \in [0, 1)$

Now, $\sqrt{n}\int_0^{1/n} e^{-nx}dx \le \sqrt{n}\int_0^{1/n} f(x)e^{-nx}dx \le \frac{\sqrt{n}}{\sin 1}\int_0^{1/n} e^{-nx}dx$

$$\Rightarrow \lim_{n\to\infty}\frac{1-\frac{1}{e}}{\sqrt{n}} \le \lim_{n\to\infty} I_n \ \frac{1-\frac{1}{e}}{(\sin 1)\sqrt{n}}$$

$$\Rightarrow 0 \le \lim_{n\to\infty} I_n \le 0$$

$$\Rightarrow \lim_{n\to\infty} I_n = 0$$

Q.37 (B)

$$\int_1^3 ((x-2)^4\sin^3(x-2) + (x-2)^{2019} + 1)dx$$

$x - 2 = t \Rightarrow dx = dt$

$$\int_{-1}^{1}(t^4\sin^3 t + t^{2019} + 1)dt = \int_{-1}^{1}dt = t]_{-1}^{1} = 2$$

Q.38 (C)

$$I_n = \frac{1}{2}\int_0^{\pi}\left(\frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} + \frac{(\pi-x)\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}\right)dx$$

$$= \frac{\pi}{2}\int_0^{\pi}\frac{\sin^{2n}x\,dx}{\sin^{2n}x+\cos^{2n}x}$$

$$= 2\times\frac{\pi}{2}\int_0^{\pi/2}\frac{\sin^{2n}x\,dx}{\sin^{2n}x+\cos^{2n}x}$$

$$= \frac{\pi}{2}\int_0^{\pi/2}\frac{\sin^{2n}x+\cos^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$$

$$= \frac{\pi}{2}\times\frac{\pi}{2} = \frac{\pi^2}{4}$$

$$\Rightarrow I_m = I_n \,\forall\, m, n$$

Q.39 (B)

$$\int_1^{\sqrt{2}+1}\frac{(x^2-1)}{x\left(x+\frac{1}{x}\right)x\sqrt{x^2+\frac{1}{x^2}}}\,dx$$

$$= \int_1^{\sqrt{2}+1}\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2-2}}\,dx$$

Let = Let $x + \frac{1}{x} = \sqrt{2}\sec\theta$

$$\left(1-\frac{1}{x^2}\right)dx = \sqrt{2}\sec\theta\tan\theta\,d\theta$$

$$\int_{\pi/4}^{\pi/3}\frac{\sqrt{2}\sec\theta\tan\theta\,d\theta}{\sqrt{2}\sec\theta\sqrt{2}\tan\theta}$$

$$= \frac{\pi}{12\sqrt{2}}$$

JEE-MAIN

PREVIOUS YEAR'S

Q.1 (2)

$$I = \int_{-1}^{0} x^2 .e^{-1}\, dx + \int_{0}^{1} x^2\, dx$$

$$\therefore I = \left.\frac{x^3}{3e}\right|_{-1}^{0} + \left.\frac{x^3}{3}\right|_{0}^{1}$$

$$\Rightarrow I = \frac{1}{3e} + \frac{1}{3}$$

Q.2 (2)

Let limit be L

$$\text{So } L = e^{\lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}{n^2}\right)} = e^k \text{ (say)}$$

Now assume $n = 2^p + A, \lambda \in \{0, 1, 2, \ldots, 2^p - 1\}$

$$\text{Now assume } 1 + \left(\frac{1}{2}+\frac{1}{3}\right) + \left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right) + \ldots +$$

$$\left(\frac{1}{s^{p-1}} + \frac{1}{2^{p-1}+1} + \ldots + \frac{1}{s^p - 1}\right)$$

$$+ \left(\frac{1}{s^p} + \frac{1}{2^p+1} + \ldots + \frac{1}{s^p - \lambda}\right) = S$$

$$\text{So } S < 1 + \left(\frac{1}{2}+\frac{1}{2}\right) + \left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) +$$

$$\ldots + \underbrace{\left(\frac{1}{2^p}+\frac{1}{2^p}+\ldots+\frac{1}{2^p}\right)}_{(\lambda+1)\text{times}}$$

$$\Rightarrow S < \underbrace{1+1+1\ldots\ldots}_{p\text{ times}} + \frac{\lambda+1}{2p} < p+1$$

$$\text{Hence } k \le \lim_{n\to\infty} \frac{p+1}{2^p} = 0$$

$$\text{Also} \quad S > \underbrace{\left(\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots\ldots+\frac{1}{n}\right)}_{n\text{ times}} = 1$$

$$\text{Hence } k \ge \lim_{n\to\infty} \frac{1}{n} = 0$$

So L = 1

Q.3 (1)

$$1 = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{n}{(n+r)^2}$$

$$\therefore \quad L = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} \frac{1}{\left(\frac{r}{n}\right)^2 + 2\left(\frac{r}{n}\right) + 1}$$

$$\therefore \quad L = \int_0^1 \frac{dx}{(x+1)^2} = \left.\frac{-1}{x+1}\right|_0^1$$

$$L = \frac{1}{2} + 1 = \frac{1}{2}$$

Q.4 (3)

$$I_n = \int_{\pi/4}^{\pi/2} (\cot x)^n x\, dx$$

$$I_n + I_{n+2} = \int_{\pi/4}^{\pi/2} ((\cot x)^n + (\cot x)^{n+2})\, dx$$

$$= \int_{\pi/4}^{\pi/2} (\cot x)^n \operatorname{cosec}^2 x\, dx$$

cot x = t

$$= -\int_1^0 t^n\, dx = \int_0^1 t^n\, dx = \left.\frac{t^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$$

$$\therefore \quad I_n + I_{n+2} = \frac{1}{n+1}$$

$$\therefore \quad I_2 + I_4 = \frac{1}{3}$$

$$I_3 + I_5 = \frac{1}{4}$$

$$I_4 + I_6 = \frac{1}{5}$$

$\Rightarrow$ $I_2 + I_4, I_3 + I_5, I_4 + I_6$ are in H.P.

Q.5 [6.33]

$$I = \int_0^2 |(x-2)(x+1)| + |(-x-2)(-x+1)|\,dx$$

$$= \int_{-2}^{2} |((-x^2 - x + 2) + (x+2)|x-1||)dx$$

$= \left(-\frac{x^3}{3}+\frac{x^2}{2}+2x\right)\Big|_0^2 +$

$\int_0^1 (x+2)(-x+1)dx + \int_1^2 (x^2+x-2)\,dx$

$= \left(-\frac{8}{3}+2+4\right) +$

$\int_0^1 (-x^2-x+2)\,dx + \left(\frac{x^3}{3}+\frac{x^2}{2}-2x\right)\Big|_0^1$

$= \frac{10}{3} + \left(-\frac{x^3}{3}-\frac{x^2}{2}-2x\right)\Big|_1^2 + \left(\frac{8}{3}+2-4\right) -$

$\left(\frac{1}{3}+\frac{1}{2}-2\right)$

$= \frac{10}{3} + \left(-\frac{1}{3}-\frac{1}{2}+2\right) + \frac{2}{3} + \frac{7}{6} = \frac{19}{3}$

Q.6 (2)

$f'(x) = f'(2-x)$

On integrating both side $f(x) = -f(2-x) + c$

put $x = 0$

$f(0) + f(2) = c \Rightarrow \quad c = 1 + e^2$

$\Rightarrow f(x) + f(2-x) = 1 + e^2 \quad \ldots\ldots(i)$

$I = \int_0^2 f(x)\,dx = \int_0^1 \{f(x) + f(2-x)\}\,dx = (1 + e^2)$

Q.7 (3)

$I = \int_1^3 -3dx + \int_1^3 [(x-1)^2]dx \quad x - 1 = t;\ dx = dt$

$I = (-6) + \int_0^2 [t^2]dt$

$I = \int_0^1 0\,dt + \int_1^{\sqrt{2}} 1\,dt + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dt + \int_{\sqrt{3}}^{2} 3\,dt$

$I = -6 + \left(\sqrt{2}-1\right) + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3}$

$I = -1 - \sqrt{2} - \sqrt{3}$

Q.8 (1)

Given $f(x)\,f''(x) - (f'(x))^2 = 0$

Let $h(x) = \frac{f(x)}{f'(x)}$

$\Rightarrow h'(x) = 0 \qquad \Rightarrow h(x) = k$

$\Rightarrow \frac{f(x)}{f'(x)} = k \qquad \Rightarrow f'(x) = k\,f'(x)$

$\Rightarrow f(x) = k\,f'(0) \qquad \Rightarrow 1 = k(2) \Rightarrow k = \frac{1}{2}$

New $f(x) = \frac{1}{2} f'(x) \qquad \Rightarrow \int 2dx = \int \frac{f'(x)}{f(x)}\,dx$

$\Rightarrow 2x = \ln|f(x)| + C$

As $f(0) = 1 \Rightarrow C = 0$

$\Rightarrow 2x = \ln|f(x)| \Rightarrow f(x) = \pm e^{2x}$

As $f(0) = 1 \Rightarrow f(x) = e^{2x} \Rightarrow f(1) = e^2$

Q.9 (1)

$I = \int_0^{\pi/2} \left(\frac{\cos^2 x}{1+3^x} + \frac{\cos^2 x}{1+3^{-x}}\right) dx$

$= \int_0^{\pi/2} \left(\frac{\cos^2 x}{1+3^x} + \frac{3^x\cos^2 x}{1+3^x}\right) dx = \int_0^{\pi/2} \cos^2 x\,dx$

$= \frac{1}{2}\int_0^{\pi/2} (1+\cos 2x)\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)_0^{\pi/2}$

$= \frac{1}{2} = \frac{\pi}{4}$

Q.10 (1)

$\sum_{n=1}^{n=100} \int_{n-1}^{n} e^{\{x\}}\,dx$

$= 100\int_0^1 e^x = 100\,(e-1)$

Q.11 [2]

$\int_0^{\pi} |\sin 2x|dx$

Here $f(2a-x) = f(x)$

$= 2\int_0^{\pi/2} (\sin 2x)dx$

$= 2\left(-\frac{\cos 2x}{2}\right)_0^{\pi/2} \qquad = 2$

Q.12 (1)

$I_{m,n} = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$ put $x = \frac{1}{y+1}$

$$I_{m,n}=\int_{\infty}^{0}\frac{y^{n-1}}{(y+1)^{m+n}}(-1)\,dy=\int_{0}^{\infty}\frac{y^{n-1}}{(y+1)^{m+n}}dy \quad .....(i)$$

Similarly $I_{m,n}=\int_0^1 x^{n-1}.(1-x)^{m-1}dx$

$$\Rightarrow I_{m,n}=\int_0^{\infty}\frac{y^{m-1}}{(y+1)^{m+n}}dy \quad ....(ii)$$

From (i) & (ii)

$$2_{I_{m,n}}=\int_0^{\infty}\frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy$$

$$\Rightarrow 2_{I_{m,n}}=\int_0^{1}\frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy+\int_1^{\infty}\frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy$$

Put $y=\frac{1}{z}$

$$\Rightarrow 2_{I_{m,n}}=\int_0^{1}\frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy+\int_1^{\infty}\frac{z^{m-1}+z^{n-1}}{(z+1)^{m+n}}dz$$

$$\Rightarrow I_{m,n}=\int_0^{1}\frac{y^{m-1}+y^{n-1}}{(y+1)^{m+1}}dy\Rightarrow \alpha=1$$

Q.13 (4)

$$f(e)+f\left(\frac{1}{e}\right)=\int_1^e\frac{\ell nt}{1+t}dt+\int_1^{1/e}\frac{\ell nt}{1+t}dt=I_1+I_2$$

$$I_2=\int_1^{1/e}\frac{\ell nt}{1+t}dt \qquad \text{put } t=\frac{1}{z}\ dt=-\frac{dz}{z^2}$$

$$\int_1^e-\frac{\ell nz}{1+\frac{1}{z}}\times\left(-\frac{dz}{z^2}\right)=\int_1^e\frac{\ell nz}{z(z+1)}dz$$

$$f(e)+f\left(\frac{1}{e}\right)=\int_1^e\frac{\ell nt}{1+t}dt+\int_1^e\frac{\ell nt}{t(t+1)}dt=\int_1^e\frac{\ell nt}{1+t}+\frac{\ell nt}{t(t+1)}dt$$

$$=\int_1^e\frac{\ell nt}{t}dt=\int_0^1 u\,du=\left.\frac{u^2}{2}\right|_0^1=\frac{1}{2}$$

Q.14 (4)

$f'(x) = e^x.f(x) + e^x$

$$\Rightarrow\frac{f'(x)}{f(x)+1}=e^x \quad \Rightarrow \ell n(f(x)+1)=e^x+c$$

put x=0

$\ell n2=1+c$

$\therefore\ \ell n(f(x)+1)=e^x+\ell n2-1$

$\Rightarrow f(x)+1=2.\,e^{e^x-1}$

$\Rightarrow f(x)=2e^{e^x-1}-1$

Q.15 [3]

$$\int_{-a}^{0}(-2x+2)\,dx\ \int_0^2(-2x+2-x)\,dx+\int_2^a(-2x-2)\,dx=$$

22

$$x^2-2x\Big|_0^{-a}+2x\Big|_2^a+x^2-2x\Big|_2^a=22$$

$a^2 + 2a + 4 + a^2 - 2a - (4 - 4) = 22$

$2a^2 = 18 \Rightarrow a = 3$

$$\int_{-3}^{3}(x+[x])\ dx=-3-2-1+1+2=-3$$

Q.16 (2)

$$\lim_{x\to0^+}\frac{\int_0^{x^2}\sin\sqrt{t}\ dt}{x^3}=\lim_{x\to0^+}\frac{(\sin|x|2x)}{3x^2}$$

$$=\lim_{x\to0^+}\left(\frac{\sin x}{x}\right)\times\frac{2}{3}=\frac{2}{3}$$

Q.17 [1]

$$E=2\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$$

$$E=2\frac{2}{\ell n2}\int_0^1\ell n\left(1\quad\tan\frac{\pi x}{4}\right)dx \qquad .....(i)$$

replacing $x\to1-x$

$$E=\frac{2}{\ell n2}\int_0^1\ell n\left(1\quad\tan\frac{\pi}{4}(1-x)\right)dx$$

$$E=\frac{2}{\ell n2}\int_0^1\ell n\left(1\quad\tan\left(\frac{\pi}{4}-\frac{\pi}{4}x\right)\right)dx$$

$$E=\frac{2}{\ell n2}\int_0^1\ell\left(\quad+\frac{1+\tan\frac{\pi}{4}x}{1+\tan\frac{\pi}{4}x}\right)dx$$

$$E=\frac{2}{\ell n2}\int_0^1\ell n\left(\frac{2}{1+\tan\frac{\pi x}{4}}\right)dx$$

$$E = \frac{2}{\ell n 2}\int_0^1\left(\ell n 2 - \ell n\left(1+\tan\frac{\pi x}{4}\right)\right)dx \quad .....(ii)$$

equation (i) + (ii)

E = 1

Q.18 [16]

$f(x) + f(x+1) = 2$

$\Rightarrow f(x)$ is periodic with period = 2

$$I_1 = \int_0^8 f(x)dx = 4\int_0^2 f(x)dx$$

$$= 4\int_0^1 (f(x) + f(1+x))\,dx \quad = 8$$

Similarly $I_2 = 2 \times 2 = 4$

$I_1 + I_2 = 16$

Q.19 (3)

$$I = \int_0^{10} [x]\cdot e^{[x]-x+1}$$

$$I = \int_0^1 0dx + \int_1^2 1\cdot e^{2-x} + \int_2^3 2\cdot e^{3-x} + ... + \int_9^{10} 9\cdot e^{10-x}dx$$

$$\Rightarrow I = \sum_{n=0}^{9}\int_n^{n+1} n\cdot e^{n+1-x}dx$$

$$= -\sum_{n=0}^{9} n\left(e^{n+1-x}\right)_n^{n+1}$$

$$= -\sum_{n=0}^{9} n\cdot(e^0 - e^1)$$

$$= (e-1)\sum_{n=0}^{9} n$$

$$= (e-1)\cdot\frac{9\cdot 10}{2}$$

$= 45(e - 1)$

Q.20 (3)

$$\int_0^1 (x + bx + c)\,dx = 1$$

$$\frac{1}{3}+\frac{b}{2}+c = 1 \quad \Rightarrow \quad \frac{b}{2}+c = \frac{2}{3}$$

3b + 6c = 4 ...(1)

P(2) = 5

4 + 2b + c = 5

2b + c = 1 ...(2)

From (1) & (2)

$$b = \frac{2}{9} \quad \& \quad c = \frac{5}{9}$$

9(b + c) = 7

Q.21 (Bονυσ)

$$g(\alpha) = \int_{\frac{\pi}{6}}^{\pi/3} \frac{\sin^\alpha x}{(\sin^\alpha x + \cos^\alpha x)} \quad .....(i)$$

$$g(\alpha) = \int_{\frac{\pi}{6}}^{\pi/3} \frac{\cos^\alpha x}{(\sin^\alpha x + \cos^\alpha x)} \quad .....(ii)$$

(1) + (2)

$$2g(\alpha) = \frac{\pi}{6}$$

$$g(\alpha) = \frac{\pi}{12}$$

Constant and even function

Due to typing mistake it must be bonus.

Q.22 **[1]**

$$I = \left(\int_0^{\sqrt{\pi/2}} [x^2] + [-\cos x]\right)dx$$

$$= \int_0^1 0\,dx + \int_1^{\sqrt{\pi/2}} dx + \int_0^{\sqrt{\pi/2}} (-1)\,dx$$

$$= \sqrt{\frac{\pi}{2}} - 1 - \sqrt{\frac{\pi}{2}} = -1$$

$\Rightarrow |I| = 1$

Q.23 (2)

$f(x) = e^{-x}\sin x$

Now, $F(x) = \int_0^x f(t)dt \Rightarrow F'(x) = f(x)$

$$I = \int_0^1 (F'(x) + f(x))e^x\,dx = \int_0^1 (f(x) + f(x))\cdot e^x\,dx$$

$$= 2\int_0^1 f(x)\cdot e^x\,dx = 2\int_0^1 e^{-x}\sin x\cdot e^x\,dx$$

$$= 2\int_0^1 \sin x\,dx$$

$$= 2\int_0^1 \sin x\,dx$$

$$I = 2\left\{1 - \left(1 - \frac{1}{\underline{|2}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|6}} + \frac{1}{\underline{|8}}.........\right)\right\}$$

$$I = 1 - \frac{2}{\lfloor 4} + \frac{2}{\lfloor 6} - \frac{2}{\lfloor 9} + ...$$

$$1 - \frac{2}{\lfloor 4} < I < 1 - \frac{2}{\lfloor 4} + \frac{2}{\lfloor 6}$$

$$\frac{11}{12} < I < \frac{331}{360}$$

$$\Rightarrow I \in \left[\frac{11}{12}, \frac{331}{360}\right]$$

$$\Rightarrow I \in \left[\frac{330}{360}, \frac{331}{360}\right]$$ **Ans. (2)**

Q.24 (1)

Let $I = \int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \int_0^{10} \frac{[\sin 2\pi x]}{e^{\{x\}}} dx$

Function $f(x) = \frac{[\sin 2\pi x]}{e^{\{x\}}}$ is periodic with period '1'

Therefore

$$I = 10\int_0^1 \frac{[\sin 2\pi x]}{e^{\{x\}}} dx$$

$$= 10\int_0^1 \frac{[\sin 2\pi x]}{e^x} dx$$

$$= 10\left(\int_0^{1/2} \frac{[\sin 2\pi x]}{e^x} dx + \int_{1/2}^1 \frac{[\sin 2\pi x]}{e^x} dx\right)$$

$$= 10\left(0 + \int_{1/2}^1 \frac{(-1)}{e^x} dx\right)$$

$$= -10\int_{1/2}^1 e^{-x} dx$$

$$= 10\left(e^{-1} - e^{-1/2}\right)$$

Now,

$10 \cdot e^{-1} - 10 \cdot e^{-1/2} = \alpha e{-1} + \beta e^{-1/2} + \gamma$ (given)

$\Rightarrow \alpha = 10, \beta = -10, \gamma = 0$

$\Rightarrow \alpha + \beta + \gamma = 0$

Q.25 [1]

$$I_n = \int_1^e x^{19} (\log|x|)^n dx$$

$$I_n = \left|(\log|x|)^{19} \frac{x^{20}}{20}\right|_1^e - \int n(\log|x|)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^{20}}{20} dx$$

$20I_n = e^{20} - nI_{n-1}$

$\therefore\ 20I_{10} = e^{20} - 10I_9$

$20I_9 = e^{20} - 9I_8$

$\Rightarrow 20I_{10} = 10I_9 + 9I_8$

$\alpha = 10, \beta = 9$

Q.26 [512]

$$I = 2\int_0^4 f(x)^2 dx \ \{\text{Even funtion}\}$$

$$2 = \int_0^4 \left(4x^3 - g(4-x)\right) dx$$

$$= 2\left(\left.\frac{4x^4}{4}\right|_0^4 - \int_0^4 g(4-x)dx\right)$$

$= 2(256 - 0) = 512$

Q.27 (3)

$\frac{1}{3} \le f(t) \le \ \forall t \in [0, 1]$

$0 \le f(t) \le \frac{1}{2} \ \forall t \in [0, 3]$

Now, $(g) = 3\int_0^3 f(t)dt = \int_0^1 f(t)\, dt + \int_1^3 f(t)\, dt$

$\because \int_0^1 \frac{1}{3} dt \le \int_0^1 f(t)\, dt \le \int_0^1 1.\, dt$ ....(1)

and $\int_1^3 0\, dt \le \int_1^3 f(1)\, dt \le \int_1^3 \frac{1}{2} dt$ ....(2)

Adding, we get

$$\frac{1}{3} + 0 \le g(3) \le 1 + \frac{1}{2}(3-1)$$

$$\frac{1}{3} \le g(3) \le 2$$

Q.28 (1)

Q.29 (3)

Q.30 (2)

Q.31 (4)

Q.32 (2)

Q.33 [1]

Q.34 [16]

Q.35 (3)

Q.36 [5]

Q.37 (2)

Q.38 [2]

Q.39 (1)

Q.40 (4)

Q.41 (2)

Q.42 (2)

$$\lim_{x\to\frac{\pi}{4}} \frac{\frac{\pi}{4}\int_{2}^{\sec^2 x} f(x)dx}{x^2-\frac{\pi^2}{16}}$$

$$\lim_{x\to\frac{\pi}{4}} \frac{\pi}{4}.\frac{\left[f(\sec^2 x).2\sec x.\sec x\tan x\right]}{2x}$$

$$\lim_{x\to\frac{\pi}{4}} \frac{\pi}{4} f(\sec^2 x).\sec^3 x.\frac{\sin x}{x}$$

$$\frac{\pi}{4} f(2).\left(\sqrt{2}\right)^3.\frac{1}{\sqrt{2}}\times\frac{4}{\pi}$$

$\Rightarrow 2f(2)$

Q.43 (4)

Q.44 (2)

Q.45 (4)

Q.46 [5]

Q.47 [4]

Q.48 (2)

Q.49 (3)

Q.50 (2)

Q.51 (2)

Q.52 (1)

Q.53 (1)

JEE-ADVANCED PREVIOUS YEAR'S

Q.1 (A)

Put $x^2 = t$

$$x\,dx = \frac{dt}{2}$$

$$\therefore I = \int_{\ell n2}^{\ell n3} \frac{\sin t}{\sin t + \sin(\ell n6 - t)} \cdot \frac{dt}{2} \quad ......(1)$$

apply $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \frac{1}{2}\int_{\ell n2}^{\ell n3} \frac{\sin(\ell n6 - t)}{\sin(\ell n6 - t) + \sin t}\, dt \quad ....(2)$$

adding (1) and (2)

$$2I = \frac{1}{2}\int_{\ell n2}^{\ell n3} 1.dt$$

$$\Rightarrow I = \frac{1}{4}\ell n\frac{3}{2}$$

Q.2 (B)

$$\int_{-\pi/2}^{\pi/2}\left(x^2 + \ell n\left(\frac{\pi+x}{\pi-x}\right)\right)\cos x\,dx = 2\int_0^{\pi/2} x^2\cos x\,dx + 0$$

$$\left(\because \ell n\left(\frac{\pi+x}{\pi-x}\right) \text{is an odd function}\right)$$

$$= 2\left[\left(x^2\sin x\right)_0^{\pi/2} - \int_0^{\pi/2} 2x\sin x\,dx\right]$$

$$= 2\left(\frac{\pi^2}{4} - 0\right) - 4\int_0^{\pi/2} x\sin x\,dx$$

$$= \frac{\pi^2}{2} - 4\left[\left(-x\cos x\right)_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx\right] = \frac{\pi^2}{2} - 4$$

Comprehension # 2 (Q. No.3 & 4)

Q.3 (B)

$f(x) = (1-x)^2\sin^2 x + x^2 : x \in R$

$$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)dt$$

$\therefore\ g'(x) = \left(\frac{2(x-1)}{x+1} - \ln x\right) f(x)\ .\ 1$

let $\phi(x) = \frac{2(x-1)}{x+1} - \ln x$

$\phi'(x) = \frac{2[(x+1)-(x-1).1]}{(x+1)^2} - \frac{1}{x}$

$= \frac{4}{(x+1)^2} - \frac{1}{x}$

$= \frac{-x^2+2x-1}{x(x+1)^2} = \frac{-(x-1)^2}{x(x+1)^2}$

$\therefore \phi'(x) \le 0$

$\therefore$ for $x \in (1, \infty), \phi(x) < 0$

$\therefore g'(x) < 0$ for $x \in (1, \infty)$

Q.4 (C)

$f(x) + 2x = (1-x)^2 \sin^2 x + x^2 + 2x$

$\because\ f(x) + 2x = 2(1 + x^2)$

$\Rightarrow (1-x)^2 \sin^2 x + x^2 + 2x = 2 + 2x^2$

$(1-x)^2 \sin^2 x = x^2 - 2x + 1 + 1$

$= (1-x)^2 + 1$

$\Rightarrow (1-x)^2 \cos^2 x = -1$

which can never be possible

P is not true

$\Rightarrow$ Let $H(x) = 2f(x) + 1 - 2x(1 + x)$

$H(0) = 2f(0) + 1 - 0 = 1$

$H(1) = 2f(1) + 1 - 4 = -3$

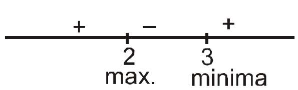
$\Rightarrow$ so H(x) has a solution

so Q is true

Q.5 (ABCD)

$f(x) = \int_0^x e^{t^2} \cdot (t-2)(t-3)\,dt$

$f'(x) = 1 \cdot e^{x^2} \cdot (x-2)(x-3)$



(i) x = 2 is local maxima

(ii) x = 3 is local minima

(iii) It is decreasing in $x \in (2, 3)$

(iv) $f''(x) = e^{x^2} \cdot (x-2) + e^{x^2}(x-3) + 2x e^{x^2}(x-2)(x-3)$

$= e^{x^2} \cdot [x - 2 + x - 3 + 2x(x-2)(x-3)]$

$f''(x) = 0$

$f''(x) = e^{x^2}(2x^3 - 10x^2 + 14x - 5)$

$f''(0) < 0$ and $f''(1) > 0$

so $f''(c) = 0$ where $c \in (0, 1)$

Q.6 (D)

$f'(x) - 2f(x) < 0$

$\frac{d}{dx}(e^{-2x} f(x)) < 0$

$\Rightarrow e^{-2x} f(x)$ is decreasing

$\Rightarrow x > 1/2$

$e^{-2x} f(x) < 1/e$

$\Rightarrow f(x) < e^{2x-1}$

$\Rightarrow 0 < \int_{1/2}^{1} f(x)\,dx < \int_{1/2}^{1} \left(e^{2x-1}\right) dx$

$\Rightarrow 0 < \int_{1/2}^{1} f(x)\,dx < \frac{e-1}{2}$

Q.7 (B)

$$\frac{2\sum r^a}{(n+1)^{a-1}\left(2n^2a + n^2 + n\right)}$$

$$\Rightarrow \frac{2\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a}{(1+1/n)^{a-1}\left(2n^2a + n^2 + n\right)} \Rightarrow \frac{2\int_0^1 x^a dx}{2a+1}$$

$$\frac{2}{(2a+1)(a+1)} = \frac{1}{60}$$

$120 = (2a+1)(a+1)$

$a = 7, -17/2$ **(–17/2 reject)**

Q.8 (AC)

It may be discontinuous at x = a or x = b

$\lim_{x \to a^-} g(x) = 0$

$\lim_{x \to a^+} g(x) = \lim_{x \to a^+} \int_a^x f(t)dt = \int_a^a f(t)dt = 0$

$g(a) = \int_a^a f(t)dt = 0$

Similarly at x = b we will get continuous

So g(x) is continuous $\forall\ x \in R$

$$g'(x) = \begin{cases} 0 & x < a \\ f(x) & a \le x \le b \\ 0 & x > b \end{cases}$$

$g'(a^-) = 0$ $\quad$ $g'(b^-) = f(b)$

$g'(a^+) = f(a)$ $\quad$ $g'(b^+) = 0$

Since f(x) co-domain is $[1, \infty)$ f(a) & f(b) can never be zero.
Hence it is non derivable at x = a & x = b.

Q.9 (ACD)

$$f(x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$$

$$f'(x) = \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \frac{x}{x^2} e^{-\left(x+\frac{1}{x}\right)}$$

$$f'(x) = \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x}$$

(A) For $x \in [1, \infty)$ $f'(x) > 0$ so (A) is correct.

(B) Obvious wrong.

(C) $f(x) + f(1/x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t} + \underbrace{\int_{x}^{1/x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt}_{\text{put } t=\frac{1}{p}}$

$$\int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t} - \int_{1/x}^{x} e^{-\left(p+\frac{1}{p}\right)} \frac{dp}{p} = 0$$

(C) is correct

(D) Since $f(x) = -f\left(\frac{1}{x}\right)$

$f(2^x) = -f\left(\frac{1}{2^x}\right)$

$f(2^x) = -f(2^{-x})$

odd.

(D) is correct

ACD is answer

Q.10 [2]

$$4x^3 . \frac{d}{dx}\left(1-x^2\right)^5\Big|_0^1 - 12\int_0^1 x^2 . \frac{d}{dx}\left(1-x^2\right)^5 dx$$

$$= -12\left[\left(x^2.\left(1-x^2\right)^5\right)_0^1 - 2\int_0^1 x.\left(1-x^2\right)^5 dx\right]$$

$$= 12\int_0^1 2x\left(1-x^2\right)^5 dx$$

$$= -12\int_1^0 t^5 dt = \frac{12}{6}\left(t^6\right)_0^1 = 2.$$

Alternative :

$$\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1-x^2)^5\right\} dx$$

$\frac{d}{dx}\left(\frac{d(1*-x^2)^5}{dx}\right) = \frac{d}{dx}(5(1 - x^2)^4 (-3x))$

$= -10 \frac{d}{dx}(x(1 - x^2)^4)$

$= -10[(1 - x^2)^4 + x^4(1 - x^2)^3 (-2x)]$

$= [-10(1 - x^2)^3 [1 - x^2 - 8x^2]$

Hence Integral

$$= -40 \int_0^1 x^3(1-x^2)^3(1-9x^2)dx$$

Put $x = \sin\theta$

$$= -40 \int_0^{\pi/2} \sin^3\theta\cos^7\theta d\theta + 360 \int_0^1 \sin^5\theta\cos^7\theta\, d\theta$$

$$= -40 . 1 . \frac{2.6.4.2}{10.8.6.4.2} + 360 . \frac{4.2.6.4.2}{12.10.8.6.4.2}$$

$= -1 + 3 = 2$ **Ans.**

Q.11 (A)

$$I = \int_{\pi/4}^{\pi/2} (2\text{cosec}x)^{17} dx$$

Put $\ell n \tan x/2 = t \Rightarrow \tan\frac{x}{2} = e^t$

$\Rightarrow \sin x = \frac{2e^t}{1+e^{2t}}$

$\text{cosec } x = \frac{e^t + e^{-t}}{2}$

$$I = 2\int_{\ell n(\sqrt{2}-1)}^{0} (e^t + e^{-t})^{16}.dt$$

$$= 2\int_{-\ell n(\sqrt{2}+1)}^{0} (e^t + e^{-t})^{16}.dt$$

since$(e^t + e^{-t})^{16}$ is an even function $\int_{-a}^{0} = \int_{0}^{a}$

Hence $I = \int_{0}^{\ell n(\sqrt{2}+1)} 2(e^t + e^{-t})^{16} dt$

Comprehension # 3 (Q. No. 12 & 13)

Q.12 (A)

$$g'(a) = \int_h^{1-h} \frac{\partial}{\partial a} t^{-a}(1-t)^{a-1}dt$$

$$= -\int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt + t^{-a}(1-t)^{a-1}\,dt = 0$$

g(a) = constant $\Rightarrow g(a) = \lambda$

$$g(a) = \underset{h\to 0^+}{\text{Lt}} \int_h^{1-h} \frac{1}{\sqrt{t(1-t)}}dt$$

$$= \int_h^{1-h} \frac{dt}{\sqrt{-\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}} = \left(\sin^{-1}\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)_h^{1-h}$$

$$= \sin^{-1}(2t-1)\Big|_h^{1-h} = \sin^{-1}(1-2h) - \sin^{-1}(2h-1) = \pi$$

Q.13 (D)

$$g(a) = \lim_{h\to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}dt$$

$$g(1-a) = \lim_{h\to 0^+} \int_h^{1-h} t^{-(1-a)}(1-t)^{(1-a)-1}dt$$

$$= \lim_{h\to 0^+} \int_h^{1-h} t^{a-1}(1-t)^{-a}dt$$

$$= \lim_{h\to 0^+} \int_h^{1-h} (1-t)^{a-1}(1-(1-t))^{-a}\,dt$$

$$\left\{\text{by} \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right\}$$

$$= \lim_{h\to 0^+} \int_h^{1-h} (1-t)^{a-1}t^{-a}\,dt$$

$g(1-a) = g(a)$

Q.14 (D)

(P) Let $f(x) = ax^2 + bx$, $a, b \in W$
(as $f(0) = 0$)

$$\int_0^1 ax^2 + bc = \frac{a}{3} + \frac{b}{2} = 1 \quad \Rightarrow 2a + 3b = 6$$

$\Rightarrow (a, b) \equiv (3, 0), (0, 2)$

Number of such polynomials = 2

(Q) $f(x) = \sqrt{2}\sin\left(x^2 + \frac{\pi}{4}\right)$

$x^2 + \frac{\pi}{4} = 2n\pi + \frac{\pi}{2}$ if f(x) is maximum

$x^2 = 2n\pi + \frac{\pi}{4}$

for n = 0, 1 $x^2 \in [0, 13]$

(R) $$\int_{-2}^{2} \frac{3x^2}{1+e^x}dx = \int_0^2 3x^2\left(\frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\right)dx$$

$$\left\{\int_{-a}^{a} f(x)dx = \int_0^a \left(f(x) + f(-x)\right)dx\right\}$$

$$= \int_0^2 3x^2\left(\frac{1}{1+e^x} + \frac{e^x}{1+e^x}\right)dx = \int_0^2 3x^2dx = x^3\Big|_0^2 = 8$$

(S) $$\int_{-1/2}^{1/2} \cos 2x\, \ell n\left(\frac{1+x}{1-x}\right)dx = 0$$

(as it is an odd function)

Hence $P \to 2, Q \to 3, R \to 1, S \to 4$. **(D) Ans.**

Q.15 [0]

$$I = \int_{-1}^{2} \frac{x[x^2]}{2+[x+1]}dx = \int_{-1}^{2} \frac{x[x^2]}{3+[x]}dx$$

$$= \int_{-1}^{0} \frac{0}{3-1}dx + \int_0^1 \frac{0}{3+0}dx + \int_1^{\sqrt{2}} \frac{x.1}{3+1}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2}\right]_1^{\sqrt{2}} = \frac{2-1}{8} = \frac{1}{4} \qquad \therefore 4I - 1 = 0$$

Q.16 [9]

$$\alpha = \int_0^1 e^{9x+3\tan^{-1}x}.\left(\frac{12+9x^2}{1+x^2}\right)dx$$

$$\Rightarrow \alpha = \left(e^{9x+3\tan^{-1}x}\right)_0^1$$

$$\Rightarrow \alpha = e^{9+\frac{3\pi}{4}} - 1$$

$$\Rightarrow \ell n(1+\alpha) = 9 + \frac{3\pi}{4}$$

Aliter :

$$\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$$

Let $9x + 3\tan^{-1}x = t$

$$\Rightarrow \left(9+\frac{3}{1+x^2}\right)dx = dt \Rightarrow \left(\frac{12+9x^2}{1+x^2}\right)dx = dt$$

$$\Rightarrow \alpha = \int_0^{9+3\pi/4} e^t dt = \left(e^t\right)_0^{9+3\pi/4} = e^{9+3\pi/4} - 1$$

Now $\log_e|1+\alpha| - 3\pi/4 = \log_e e^{(9+3\pi/4)} - 3\pi/4 = 9$

Q.17 [7]

$$F(x) = \int_{-1}^{x} f(t)dt = \int_{1}^{x} f(t)dt$$

$$G(x) = \int_{-1}^{x} t|f(f(t))|dt = \int_{-1}^{x} t|f(f(t))|dt$$

$$\lim_{x\to 1}\frac{F(x)}{G(x)}$$

L' hospitals $\displaystyle\lim_{x\to 1}\frac{f(x)}{x\,|f(f(x))|} = \frac{1}{14}$

$$\frac{\frac{1}{2}}{1\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14}$$

$$f\left(\frac{1}{2}\right) = 7$$

Q.18 (A,B)

$f(x) = (7\tan^6 x - 3\tan^2 x).\sec^2 x$

$$\therefore \int_0^{\frac{\pi}{4}} f(x)\,dx = \int_0^1 \left(7t^6 - 3t^2\right)dt = (t^7 - t^3)_0^1 = 0$$

$$\text{Now } \int_0^{\frac{\pi}{4}} xf(x)dx = \int_0^1 \underbrace{\left(7t^6-3t^2\right)}_{II}\underbrace{\tan^{-1}t}_{I}\,dt$$

$$= \left(\tan^{-1}t.\left(t^7 - t^3\right)\right)_0^1 - \int_0^1 \left(t^7 - t^3\right).\frac{1}{1+t^2}dt$$

$$= \int_0^1 \frac{t^3\left(1-t^4\right)}{1+t^2}dt = \int_0^1 t^3\left(1-t^2\right)dt$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

Q.19 (D)

$$f'(x) = \frac{192x^3}{2+\sin^4(\pi x)}\ \forall x\in R;\ f\left(\frac{1}{2}\right) = 0$$

Now $64x^3 \le f'(x) \le 96x^3\ \forall x \in \left[\frac{1}{2},1\right]$

So $16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2}\ \forall x \in \left[\frac{1}{2},1\right]$

$$\frac{16}{5}.\frac{31}{32} - \frac{1}{2} \le \int_{1/2}^{1} f(x)dx \le \frac{24}{5}.\frac{31}{32} - \frac{3}{4} \Rightarrow \frac{26}{10} \le$$

$$\int_{1/2}^{1} f(x)dx \le \frac{78}{20} \text{ hence (D)}$$

(A) is incorrect as $\displaystyle\int_{1/2}^{1} f(x)dx \le \frac{78}{20}$

(B) is incorrect as $\displaystyle\int_{1/2}^{1} f(x)dx > \frac{26}{10}$

(C) is incorrect as $\displaystyle\int_{1/2}^{1} f(x)dx > 0$

Q.20 (A,C)

$$I_1 = \int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt + \int_{\pi}^{2\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt +$$

$$\int_{2\pi}^{3\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt + \int_{3\pi}^{4\pi}\left(\sin^6 at + \cos^4 at\right)dt$$

$$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi})\int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt$$

$$\Rightarrow \frac{I_1}{I_2} = 1 + e^{\pi} + e^{2\pi} + e^{3\pi} = \frac{e^{4\pi}-1}{e^{\pi}-1}$$

Comprehension # 4 (Q. No. 21 & 22)

Q.21 (A,B,C)

$f'(x) = xf'(x) + f(x)$

$\Rightarrow f'(1) = f'(1) + f(1) = f'(1) < 0$

$\Rightarrow$ (A)

$f(2) = 2f(2) < 0$

$\Rightarrow$ (B)

for $x \in (1, 3)$ $f'(x) = xf'(x) + f(x) < 0$

$\Rightarrow$ (C)

Q.22 (C,D)

$\int_1^3 x^3 f''(x)dx = 40 \Rightarrow \left[x^3 f'(x)\right]_1^3 - \int_1^3 3x^2 f'(x)dx = 40$

$\Rightarrow \left[x^2 f'(x) - x^2 F(x)\right]_1^3 - 3(-12) = 40$

$\Rightarrow 9f'(3) - 9F(3) - f'(1) + f(1) = 4$

$\Rightarrow 9f'(3) + 36 - f'(1) + 0 = 4$

$\Rightarrow 9f'(3) - f'(1) + 32 = 0 \quad \Rightarrow$ (C)

$\Rightarrow \int_1^3 x^2 F'(x)dx = -12 \Rightarrow \left[x^2 F(x)\right]_1^3 - \int_1^3 2xF(x)dx = -12$

$\Rightarrow -36 - 2\int_1^3 f(x)dx = -12 \Rightarrow \int_1^3 f(x)dx = -12$

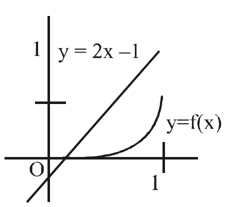
$\Rightarrow$ (D)

Q.23 [1]

$$f'(x) = \frac{x^2}{1+x^4} > 0$$

$\Rightarrow f(x)$ is monotonically increasing

Now, $f(0) = 0$ & $f(1) = \int_0^1 \frac{t^2}{1+t^4}dt$



$\because 0 < \frac{x^2}{1+x^4} < 1$, $\therefore 0 < \int_0^1 \frac{t^2}{1+t^4}dt < \int_0^1 1dt$

$$\Rightarrow 0 < \int_0^1 \frac{t^2}{1+t^4}dt < 1$$

Hence, eq^n has one solution in (0, 1].

Ans- 1

Q.24 (A)

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x}dx$$

$$Now, I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^{-x}}dx$$

$$\therefore 2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x(1+e^x)}{1+e^x}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x\, dx$$

$$\Rightarrow 2I = 2\int_0^{\frac{\pi}{2}} x^2 \cos x\, dx$$

$$\Rightarrow I = \left(x^2 \sin x - 2x(-\cos x) + 2(-\sin x)\right)_0^{\frac{\pi}{2}} = \frac{\pi^2}{4} - 2$$

Q.25 (B, C)

$$f(x) \lim_{n\to\infty}\left(\frac{n^n . n^n\left(\frac{x}{n}+1\right)\left(\frac{x}{n}+\frac{1}{2}\right)\left(\frac{x}{n}+\frac{1}{3}\right)\ldots\ldots\left(\frac{x}{n}+\frac{1}{n}\right)}{n!.(n^2)^n\left(\frac{x^2}{n^2}+1\right)\left(\frac{x^2}{n^2}+\frac{1}{2^2}\right)\ldots\ldots\left(\frac{x^2}{n^2}+\frac{1}{n^2}\right)}\right)^{\frac{x}{n}}$$

$$\ln f(x) = \frac{x}{n}\lim_{n\to\infty}\ln\left[\frac{\left(\frac{x}{n}+1\right)\left(\frac{x}{n}+\frac{1}{2}\right)\left(\frac{x}{n}+\frac{1}{3}\right)\ldots\left(\frac{x}{n}+\frac{1}{n}\right)}{\left(\frac{1.x^2}{n^2}+1\right)\left(\frac{2.x^2}{n^2}+\frac{1}{2}\right)\left(\frac{3.x^2}{n^2}+\frac{1}{3}\right)\ldots\ldots\left(\frac{n.x^2}{n^2}+\frac{1}{n}\right)}\right]$$

$$= \frac{x}{n}\lim_{n\to\infty}\sum_{r=1}^{n}\ln\left(\frac{\frac{x}{n}+\frac{1}{r}}{\frac{rx^2}{n^2}+\frac{1}{r}}\right) = \frac{x}{n}\lim_{n\to 0}\sum_{r=1}^{n}\ln\left(\frac{x+\frac{n}{x}}{\frac{rx^2}{n}+\frac{r}{r}}\right)$$

$$= x\int_0^1 \ln\left(\frac{x+\frac{1}{t}}{tx^2+\frac{1}{t}}\right)dt = x\int_0^1 \ln(xt+1)\,dt - x\int_0^1 \ln\left(x^2t^2+1\right)dt$$

$$Let\ tx = u \qquad \Rightarrow dt = \frac{du}{x}$$

$$\Rightarrow \ln\ f(x) = \int_0^x \ln(1+u)\,du - \int_0^x \ln\left(1+u^2\right)du$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ln\left(\frac{1+x}{1+x^2}\right)$$

$$\frac{f'(2)}{f(2)}=\ln\left(\frac{3}{5}\right)<0 \Rightarrow f'(2)<0$$

$$\frac{f'(3)}{f(3)}=\ln\left(\frac{2}{5}\right) \Rightarrow \frac{f'(2)}{f(2)} \ge \frac{f'(3)}{f(3)}$$

Now, $\frac{f'(x)}{f(x)}>0$ in $(0,1)$ and $\frac{f'(x)}{f(x)}<0$ in $(1,\infty)$

$\Rightarrow f'(x)>0$ in $(0,1)$ and $f'(x)<0$ in $(1,\infty)$

$\Rightarrow f(x)$ is M.I. in $(0,1)$ and $f(x)$ is M.D. in $(1,\infty)$

So, $f(1)\ge f\left(\frac{1}{2}\right)$ and $f\left(\frac{1}{3}\right)\le f\left(\frac{2}{3}\right)$

Q.26 [2]

$$g(x)=\int_x^{\pi/2}\frac{d}{dt}\left(f(t)\operatorname{cosec} t\right)dt$$

$$g(x)=f\left(\frac{\pi}{2}\right)\operatorname{cosec}\left(\frac{\pi}{2}\right)-f(x)\operatorname{cosec}x$$

$g(x)=3-f(x)\operatorname{cosec}x$

$$g(x)=3-\frac{f(x)}{\sin x}$$

$$\lim_{x\to 0}g(x)=3-\lim_{x\to 0}\frac{f(x)}{\sin x}$$

$$=3-\lim_{x\to 0}\frac{f'(x)}{\cos x}=3-\frac{1}{1}=2$$

Q.27 (BD)

Put $x-k=p$

$$I=\sum_{k=1}^{98}\int_0^1\frac{k+1}{(k+p)(k+p+1)}dp$$

$$I>\sum_{k=1}^{98}\int_0^1\frac{k+1}{(k+p+1)^2}dp$$

$$I>\sum_{k=1}^{98}(k+1)\left(\frac{-1}{(k+p+1)}\right)_0^1$$

$$I>\sum_{k=1}^{98}(k+1)\left(\frac{1}{k+1}-\frac{1}{k+2}\right)$$

$$I>\sum_{k=1}^{98}\frac{1}{k+2}=\frac{1}{3}+..........+\frac{1}{100}$$

$$I>\frac{1}{100}+......+\frac{1}{100}=\frac{98}{100}$$

$$I>\frac{49}{50}$$

$$\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx$$

$\frac{k+1}{x(x+1)}<\frac{k+1}{x(k+1)}$ ($\because$ least value of x + 1 is k + 1)

$$\Rightarrow \frac{k+1}{x(x+1)}<\frac{1}{x}$$

$$\Rightarrow I<\sum_{k=1}^{98}\int_k^{k+1}\frac{1}{x}dx$$

$$\Rightarrow I<\sum_{k=1}^{98}\ell n(k+1)-\ell nk \Rightarrow I<\ell n 99$$

Q.28 (BONUS)

$$g(x)=\int_{\sin x}^{\sin 2x}\sin^{-1}(t)\,dt$$

$g'(x) = \sin^{-1}(\sin 2x) \cdot \cos 2x \cdot 2 - \sin^{-1}(\sin x) \cdot \cos x$

$= 2\cos 2x \cdot \sin^{-1}(\sin 2x) - \cos x \cdot \sin^{-1}(\sin x)$

$$g'\left(-\frac{\pi}{2}\right)=2\cos(-\pi)\sin^{-1}(\sin(-\pi))$$

$$-\cos\left(-\frac{\pi}{2}\right).\sin^{-1}(\sin\left(-\frac{\pi}{2}\right))=0$$

$$g'\left(\frac{\pi}{2}\right)=2\cos(\pi)\sin^{-1}(\sin(\pi))$$

$$-\cos\left(\frac{\pi}{2}\right).\sin^{-1}(\sin\left(\frac{\pi}{2}\right))=0$$

Q.29 [1]

$$y_n=\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right).......\left(1+\frac{n}{n}\right)\right\}^{\frac{1}{n}}$$

$$y_n = \prod_{r=1}^{n}\left(1+\frac{r}{n}\right)^{1/n}$$

$$\log y_n = \frac{1}{n}\sum_{r=1}^{n}\ell n\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \lim_{n\to\infty}\log y_n = \lim_{x\to\infty}\sum_{r=1}^{n}\frac{1}{n}\ell n\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \log L = \int_0^1 \ell n(1+x)\,dx$$

$$\Rightarrow \log L = \log\frac{4}{e}$$

$$\Rightarrow L = \frac{4}{e}$$

$$\Rightarrow [L] = 1$$

Q.30 [2]

$$\int_0^{\frac{1}{2}}\frac{\left(1+\sqrt{3}\right)dx}{\left[(1+x)^2(1-x)^6\right]^{1/4}}$$

$$\int_0^{\frac{1}{2}}\frac{\left(1+\sqrt{3}\right)dx}{(1+x)^2\left[\frac{(1-x)^6}{(1+x)^6}\right]^{1/4}}$$

Put $\frac{1-x}{1+x} = t \Rightarrow \frac{=-2}{(1+x)^2} = dt$

$$I = \int_1^{1/3}\frac{\left(1+\sqrt{3}\right)dt}{-2t^{6/4}} = \frac{-\left(1+\sqrt{3}\right)}{2}\times\left|\frac{-2}{\sqrt{t}}\right|_1^{1/3} = \left(1+\sqrt{3}\right)\left(\sqrt{3}-1\right) = 2$$

Q.31 [4.00]

$$2I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\left[\frac{1}{(1+e^{\sin x})(2-\cos 2x)}+\frac{1}{(1+e^{-\sin x})(2-\cos 2x)}\right]dx$$

(using King's Rule)

$$\Rightarrow I = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4}\frac{dx}{2-\cos 2x} = \frac{2}{\pi}\int_0^{\pi/4}\frac{\sec^2 x dx}{1+3\tan^2 x}$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}\tan x)\right]_0^{\pi/4} = \frac{2}{3\sqrt{3}}$$

$$\Rightarrow 271^2 = 27\times\frac{4}{27} = 4$$

Q.32 (1,2)

$$\lim_{x\to\infty}\frac{n^{1/3}\left(\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}\right)}{n^{7/3}\left(\sum_{r=1}^{n}\frac{1}{(an+r)^2}\right)} = 54$$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{\frac{1}{n}\left(\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}\right)}{\frac{1}{n}\left(\sum_{r=1}^{n}\frac{1}{(a+r/n)^2}\right)}\right) = 54$$

$$\Rightarrow \frac{\int_0^1 x^{1/3}dx}{\int_0^1\frac{1}{(a+x)^2}dx} = 54 \quad \Rightarrow \quad \frac{\frac{3}{4}}{\frac{1}{a(a+1)}} = 54$$

$\Rightarrow a(a+1)=72 \Rightarrow a^2 + a - 72 = 0 \Rightarrow a = -9, 8$

Q.33 [0.50]

$$I = \int_0^{\pi/2}\frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)}d\theta$$

$$= \int_0^{\pi/2}\frac{3\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sec\theta}\right)}d\theta$$

$$2I = \int_0^{\pi/2}\frac{3d\theta}{\left(\sqrt{\cos\theta}+\sqrt{\sec\theta}\right)^4}$$

$$= 3\int_0^{\pi/2}\frac{\sec^2\theta d\theta}{\left(1+\sqrt{\tan\theta}\right)^4}$$

Let $1+\sqrt{\tan\theta} = t$

$$\frac{\sec^2\theta}{2\sqrt{\tan\theta}}\,d\theta = dt$$

$\sec^2\theta d\theta = 2(t-1)dt$

$$= 3\int_1^{\infty}\frac{2(t-1)dt}{t^4} = 6\int_1^{\infty}\left(t^{-3}-t^{-4}\right)dt$$

$$2I = 6\left(\frac{t^{-2}}{-2}-\frac{t^{-3}}{-3}\right)_1 = 6\left[0-0-\left\{-\frac{1}{2}+\frac{1}{3}\right\}\right]$$

I = 0.50

Q.34 (A,B,D)

(A) $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-....$

$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-....$

$\cos x \ge 1-\frac{x^2}{2}$

$\int_0^1 x\cos x \ge \int_0^1 x\left(1-\frac{x^2}{2}\right)=\frac{1}{2}-\frac{1}{8}$

$\int_0^1 x\cos x \ge \frac{3}{8}(\text{True})$

(B) $\sin x \ge x-\frac{x^3}{6}$

$\int_0^1 x\sin x \ge \int_0^1 x\left(x-\frac{x^3}{6}\right)dx$

$\int_0^1 x\sin x \ge \frac{1}{3}-\frac{1}{30} \Rightarrow \int_0^1 x\sin x\,dx \ge \frac{3}{8}$ (True)

(D) $\int_0^1 x^2\sin x\,dx \ge \int_0^1 x^2\left(x-\frac{x^3}{6}\right)dx$

$\int_0^1 x^2\sin x\,dx \ge \frac{1}{4}-\frac{1}{36}$

$\int_0^1 x^2\sin x\,dx \ge \frac{2}{9}$ (True)

(C) $\cos x < 1$

$x^2\cos x < x^2$

$\int_0^1 x^2\cos x\,dx < \int_0^1 x^2\,dx$

$\int_0^1 x^2\cos x\,dx < \frac{1}{3}$

So option 'C' is incorrect.

Q.35 [4.00]

$F(x)=\int_0^x f(t).dt$

$\Rightarrow F'(x)=f(x)$

$I=\int_0^\pi f'(x).\cos x\,dx+\int_0^\pi F(x)\cos(x)dx=2$ ...(1)

$I_1=\int_0^\pi f'(x).\cos x\,dx\,(\text{Let})$

Using by parts

$I_1=\left(\cos x.f(x)\right)_0^\pi+\int_0^\pi \sin x.f(x)dx$

$I_1=6-f(0)+\int_0^\pi \sin x.F'(x)dx$

$I_1=6-f(0)+I_2$ ...(2)

$I_2=\int_0^\pi \sin x.F'(x).dx$

Using by part we get

$I_2=\left(\sin x.F(x)\right)_0^\pi-\int_0^\pi \cos x.F(x)dx$

$I_2=-\int_0^\pi \cos x.F(x)dx$

$(2)\Rightarrow I_1=6-f(0)-\int_0^\pi \cos x.F(x)dx$

$(1)\Rightarrow I=6-f(0)=2\Rightarrow f(0)=4$

Q.36 (A,B,C)

(A) Let $g(x)=f(x)-3\cos 3x$

Now $\int_0^{\pi/3} g(x)dx=\int_0^{\pi/3} f(x)dx-3\int_0^{\pi/3}\cos 3x dx=0$

hence $g(x)=0$ has a root in $\left(0,\frac{\pi}{3}\right)$

(B) Let $h(x)=f(x)-3\sin 3x+\frac{6}{\pi}$

Now $\int_0^{\pi/3} h(x)dx=\int_0^{\pi/3} f(x)dx-3\int_0^{\pi/3}\sin 3x dx+\int_0^{\pi/3}\frac{6}{\pi}dx$

$=0-2+2=0$

Hence $h(x)=0$ has a root in $\left(0,\frac{\pi}{3}\right)$

(C) $\lim_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=\lim_{x\to 0}\underbrace{\left(\frac{x^2}{1-e^{x^2}}\right)}_{-1}\underbrace{\frac{\int_0^x f(t)dt}{x}}_{\text{Apply:L'Hopital's Rule}}$

$=-1\lim_{x\to 0}\frac{f(x)}{1}=-1$

(D) $\lim_{x\to 0}\frac{(\sin x)\int_0 f(t)dt}{x^2}$

$=\lim_{x\to 0}\underbrace{\left(\frac{\sin x}{x}\right)}_{1}\underbrace{\frac{\int_0^x f(t)dt}{x}}_{\text{Apply L'Hopitals Rule}}=1\lim_{x\to 0}\frac{f(x)}{1}=1$

Q.37 [2.00]

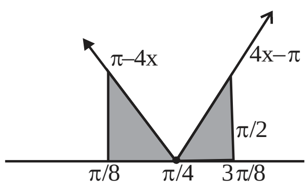
$$S_1 = \int_{\pi/8}^{3\pi} f(x)dx = \int_{\pi/8}^{3\pi/8} \sin^2 x dx = \int_{\pi/8}^{3\pi/8} \sin^2\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right)$$

$$dx = \int_{\pi/8}^{3\pi/8} \cos^2 x dx$$

$$2S_1 = \int_{\pi/8}^{3\pi/8} (\sin^2 x + \cos^2 x)dx = \frac{3\pi}{8}-\frac{\pi}{8} = \frac{\pi}{4}$$

$$\Rightarrow \frac{16S_1}{\pi} = 2$$

Q.38 [1.50]



$$S_2 = \int_{\pi/8}^{3\pi/8} f(x)g_2(x)dx = \int_{\pi/8}^{3\pi/8} \sin^2 x\,|4x-\pi|\,dx$$

$$= \int_{\pi/8}^{3\pi/8} f(x)g_2(x)dx = \int_{\pi/8}^{3\pi/8} \sin^2 x\,|4x-\pi|\,dx$$

$$= \int_{\pi/8}^{3\pi/8} (\cos^2 x)\,|\pi-4x|\,dx$$

$$\Rightarrow 2S_1 = \int_{\pi/8}^{3\pi/8} |4x-\pi|(\sin^2 x+\cos^2 x)dx = \int_{\pi/8}^{3\pi/8} |4x-\pi|\,dx$$

$$= 2\times\frac{1}{2}\times\frac{\pi}{8}\times\frac{\pi}{2} = \frac{\pi^2}{16}$$

$$\Rightarrow \frac{48S_2}{\pi^2}\times\frac{3}{2} = 1.5$$

Area Under Curve

EXERCISES

ELEMENTRY

Q.1 (3)

Given curve $y = \log x$ and $x = 1$, $x = 2$.

Hence required area = $\int_1^2 \log x\,dx = (x\log x - x)_1^2 =$

$2\log 2 - 1 = (\log 4 - 1)$ *sq. unit.*

Q.2 (3)

Required area = $\int_1^4 x\,dy = \int_1^4 \frac{\sqrt{y}}{2}dy$

$= \frac{1}{2}.\frac{2}{3}|\,y^{3/2}\,|_1^4 = \frac{7}{3}$ *sq. unit.*

Q.3 (2)

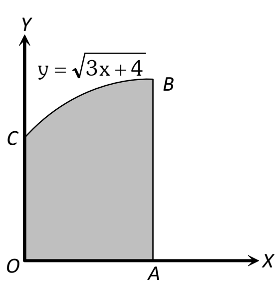
Required area $= \int_0^{\pi/4}(\sin 2x + \cos 2x)dx$

$= \left[-\frac{\cos 2x}{2} + \frac{\sin 2x}{2}\right]_0^{\pi/4}$

$= \frac{1}{2}\left[-\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + \cos 0 - \sin 0\right] = 1$sq. *unit.*

Q.4 (4)

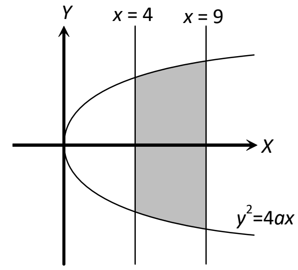
Area $= \int_0^4 \sqrt{3x+4}\,dx = \left|\frac{(3x+4)^{3/2}}{3.(3/2)}\right|_0^4$



$= \frac{2}{9} \times 56 = \frac{112}{9}$ sq. *unit.*

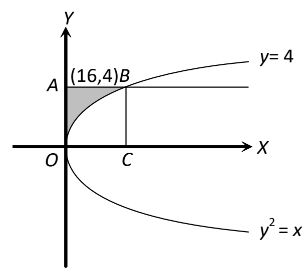
Q.5 (4)

Shaded area $A = 2\int_4^9 \sqrt{4ax}\,dx$



Q.6 (2)

Required area = area of *OABC* – area of *OBC*



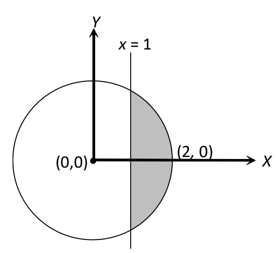
$= 16 \times 4 - \int_0^{16}\sqrt{x}\,dx = 64 - \left[\frac{x^{3/2}}{3/2}\right]_0^{16} = \frac{64}{3}.$

Q.7 (1)

required area is 1

Q.8 (1)

Area of smaller part $= 2\int_1^2 \sqrt{4 - x^2}\,dx$



$= 2\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_1^2 = 2\left[2.\frac{\pi}{2} - \left[\frac{\sqrt{3}}{2} + 2.\frac{\pi}{6}\right]\right]$

$= 2\left[\pi - \left[\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right]\right] = \frac{4\pi}{3} - \sqrt{3}$

Q.9 (2)

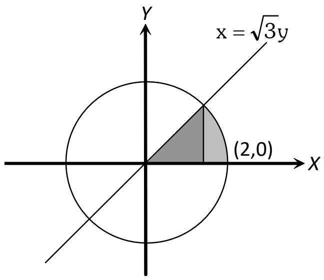
Required area

$$A=\int_0^{\pi/2}\sin^2 x.dx=\int_0^{\pi/2}\left(\frac{1-\cos 2x}{2}\right)dxa$$

$$=\frac{1}{2}[x]_0^{\pi/2}-\frac{1}{4}[\sin 2x]_0^{\pi/2}=\frac{\pi}{4}$$

Q.10 (3)

Required area $=\int_0^{\sqrt{3}}\frac{x}{\sqrt{3}}dx+\int_{\sqrt{3}}^{2}\sqrt{4-x^2}\,dx$



$$=\frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}}+\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2}$$

$$=\frac{\sqrt{3}}{2}+\left[\pi-\frac{\sqrt{3}}{2}-\frac{2\pi}{3}\right]=\frac{\pi}{3}$$

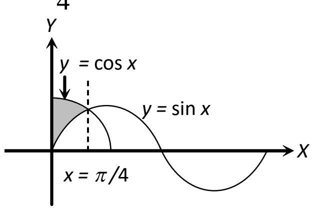
Trick : Area of sector made by an arc $=\frac{\theta^c R^2}{2}$

$$=\frac{\pi}{6}.\frac{4}{2}=\frac{\pi}{3}$$

Q.11 (1)

Given required area has been shown in the figure.

$x=\frac{\pi}{4}$ is the point of intersection of both curve



$\therefore$ Required area $=\int_0^{\pi/4}(\cos x-\sin x)\,dx$

$$=[\sin x+\cos x]_0^{\pi/4}=\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right]$$

$$=\frac{2}{\sqrt{2}}-1=\sqrt{2}-1$$

Q.12 (3)

We have $y=4x-x^2$ and $y=0$; $\therefore$ $x=0$, 4

Required area $=\int_0^4(4x-x^2)dx=\left[\frac{4x^2}{2}-\frac{x^3}{3}\right]_0^4$

$$=32-\frac{64}{3}=\frac{32}{3}\ sq.\ unit.$$

Q.13 (3)

required area is $\log\sqrt{2}-\frac{1}{4}$

Q.14 (1)

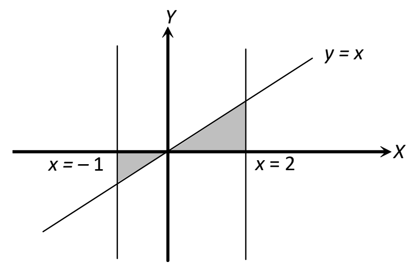
Solving $y=0$ and $y=4+3x-x^2$, we get $x=-1,4$.

Curve does not intersect *x*-axis between x = –1 and x = 4.

$\therefore$ Area $=\int_{-1}^{4}(4+3x-x^2)dx=\frac{125}{6}$.

Q.15 (4)

Bounded area $=\left|\int_{-1}^{0}x\,dx\right|+\left|\int_0^2 x\,dx\right|$



$$=\left|-\frac{1}{2}\right|+|2|=2+\frac{1}{2}=\frac{5}{2}$$

Q.16 (3)

We have $y^2=4ax\Rightarrow y=2\sqrt{ax}$

We know the equations of lines $x=a$ and $x=4a$

$\therefore$ The area inside the parabola between the lines

$$A=2\int_a^{4a}y\,dx=2\int_a^{4a}2\sqrt{ax}\,dx=4\sqrt{a}\int_a^{4a}x^{\frac{1}{2}}dx=4\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_a^{4a}$$

$$=\frac{8}{3}a^{\frac{1}{2}}\left[(4a)^{\frac{3}{2}}-(a)^{\frac{3}{2}}\right]=\frac{8}{3}a^{\frac{1}{2}}a^{\frac{3}{2}}[8-1]\ =\frac{56}{3}a^2$$

Q.17 (2)

Given, $y=-x^2+2x+3$ and $y=0$

Therefore, $x=-1$ and $x=3$

$\therefore$ Required area $=\int_{-1}^{3}(-x^2+2x+3)dx$

$$=\left[-\frac{x^3}{3}+x^2+3x\right]_{-1}^{3}=\frac{32}{3}$$

Q.18 (3)

Given curves are, $y = x^3$ and $y = \sqrt{x}$

On solving, we get $x = 0, x = 1$

Therefore, required area $= \int_0^1 (x^3 - \sqrt{x})\,dx$

$= \left[\frac{x^4}{4} - \frac{2x\sqrt{x}}{3}\right]_0^1 = \left[\frac{1}{4} - \frac{2}{3}\right] = \frac{5}{12}$,(Area can't be negative).

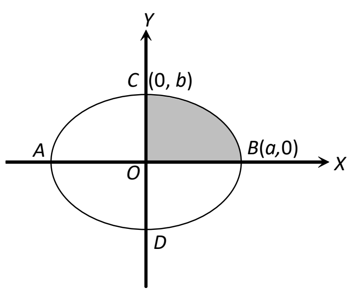
Q.19 (1)

The parabola meets x-axis at the points, where $\frac{3}{a}(a^2 - x^2) = 0 \Rightarrow x = \pm a.$ So the required area

$= \int_{-a}^{a} \frac{3}{a}(a^2 - x^2)dx = \frac{6}{a}\int_0^a (a^2 - x^2)dx = 4a^2$ *sq. unit*.

Q.20 (1)

Since the given equation contains only even powers of *x* and only even powers of *y*, the curve is symmetrical
about *y*-axis as well as *x*-axis.



$\therefore$ Whole area of given ellipse

$= 4(\text{area of BCO}) = 4 \times \int_0^a y\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$

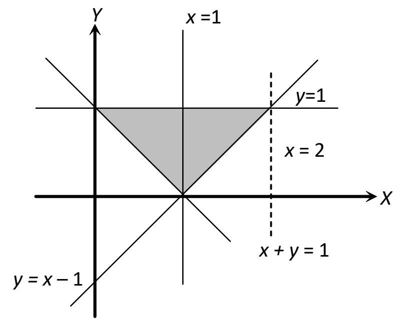
$= 4ab\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$, {Putting $x = a\sin\theta$ }

$= 2ab\left(\int_0^{\pi/2} d\theta + \int_0^{\pi/2}\cos 2\theta\, d\theta\right)$

$= [\theta]_0^{\pi/2} + \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \pi ab$ *sq. unit.*

Q.21 (2)

$y = x - 1$, if $x > 1$ and $y = -(x - 1)$, if $x < 1$



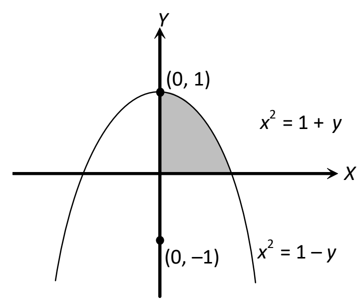
Area

$= \int_0^1 (1-x)dx + \int_1^2 (x-1)dx = \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2$

$= \left[1 - \frac{1}{2}\right] + \left[-\left(\frac{1}{2} - 1\right)\right] = \frac{1}{2} + \frac{1}{2} = 1$

Q.22 (4)

Given parabolas are $x^2 = 1 + y$, $x^2 = 1 - y$



Required area $= 4\int_0^1 (1 - x^2)\,dx = 4\left[x - \frac{x^3}{3}\right]_0^1 = \frac{8}{3}$

Q.23 (2)

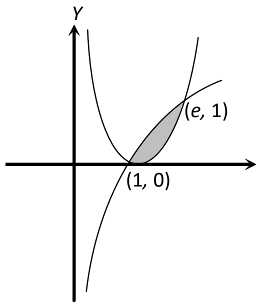
$y^2 = 8x$ and $y = x \Rightarrow x^2 = 8x \Rightarrow x = 0, 8$

$\therefore$ Required area $= \int_0^8 (2\sqrt{2}\sqrt{x} - x)dx$

$= \left[\frac{4\sqrt{2}}{3}x^{3/2} - \frac{x^2}{2}\right]_0^8 = \frac{128}{3} - \frac{64}{2} = \frac{32}{3}$ sq. *unit.*

Q.24 (1)

Required area $= \int_1^e [\log x - (\log x)^2]\,dx$



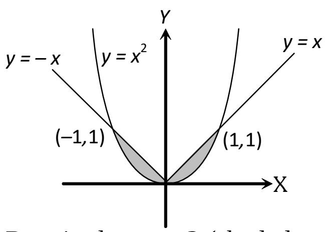
$$A=\int_1^e \log x\,dx-\int_1^e(\log x)^2\,dx$$

$$=[x\log x-x]_1^e-[x(\log x)^2-2x\log x+2x]_1^e$$

$$=[e-e-(-1)]-[e(1)^2-2e+2e-(2)]$$

$$=(1)-(e-2)=3-e$$

Q.25 (2)



Required area = 2 (shaded area in first quadrant)

$$=2\int_0^1(x-x^2)\,dx=2\times\frac{1}{6}=\frac{1}{3}$$

Q.26 (3) Given equations of curves $y=\cos x$ and $y=\sin x$ and ordinates $x=0$ to $x=\frac{\pi}{4}$. We know that area bounded by the curves

$$=\int_{x_1}^{x_2}y\,dx=\int_0^{\pi/4}\cos x\,dx-\int_0^{\pi/4}\sin x\,dx$$

$$=[\sin x]_0^{\pi/4}-[-\cos x]_0^{\pi/4}$$

$$=\left(\sin\frac{\pi}{4}-\sin 0\right)+\left(\cos\frac{\pi}{4}-\cos 0\right)=\left(\frac{1}{\sqrt{2}}-0\right)+\left(\frac{1}{\sqrt{2}}-1\right)$$

$$=\sqrt{2}-1$$

Q.27 (1)

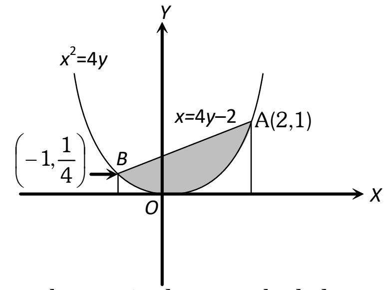
Area of the circle in first quadrant is $\frac{\pi(\pi^2)}{4}$ *i.e.*, $\frac{\pi^3}{4}$.

Also area bounded by curve $y=\sin x$ and x-axis is 2 *sq*.

unit. Hence required area is $\frac{\pi^3}{4}-2=\frac{\pi^3-8}{4}$.

Q.28 (2)

Solving the equations $x^2=4y$ and $x=4y-2$ simultaneously. The points of intersection of the parabola and the line are $A(2,1)$ and $B\left(-1,\frac{1}{4}\right)$



∴ The required area = shaded area

$$=\left[\int_{-1}^{2}y\,dx_{(\text{for }x=4y-2)}\right]-\left[\int_{-1}^{2}y\,dx_{(\text{for }x^2=4y)}\right]$$

$$=\int_{-1}^{2}\frac{1}{4}(x+2)\,dx-\int_{-1}^{2}\frac{1}{4}x^2\,dx$$

$$=\frac{1}{4}\left[\frac{x^2}{2}+2x\right]_{-1}^{2}-\frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2}=\frac{9}{8}\ \textit{sq. unit.}$$

Q.29 (2)

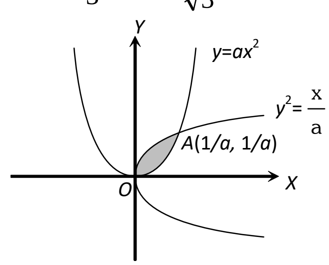
The x-coordinate of A is $\frac{1}{a}$

According to the given condition,

$$1=\int_0^{1/a}\left(\sqrt{\frac{x}{a}}-ax^2\right)dx$$

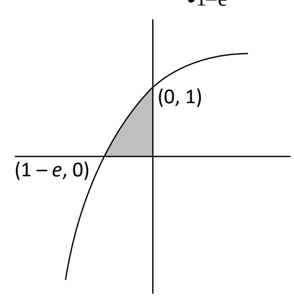
$$\Rightarrow 1=\frac{1}{\sqrt{a}}\cdot\frac{2}{3}[x^{3/2}]_0^{1/a}-\frac{a}{3}[x^3]_0^{1/a}$$

$$\Rightarrow a^2=\frac{1}{3}\Rightarrow a=\frac{1}{\sqrt{3}}$$



Q.30 (3)

Required area $=\int_{1-e}^{0}\log_e(x+e)\,dx$



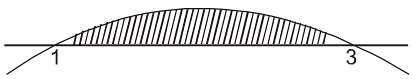
$= \int_1^e \log t \, dt = [t \log t - t]_1^e = 1$ sq. unit,

(Put $x + e = t$).

JEE-MAIN
OBJECTIVE QUESTIONS

Q.1 (3)

$y = 0 \Rightarrow x = 1, 3$



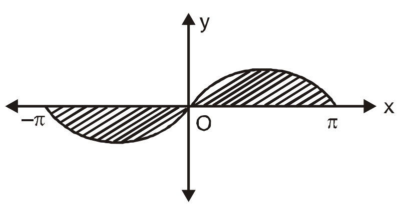
Graph of $y = 4x - x^2 - 3$

Area $= \int_1^3 (4x - x^2 - 3)dx = \frac{4}{3}$

Q.2 (1)

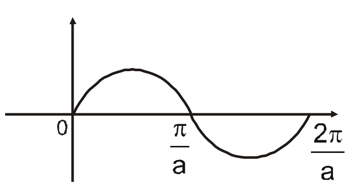
Area of bounded region $= 2\int_0^{\pi} \sin x \, dx =$

$2[-\cos x]_0^{\pi} = 2\,[1 - (-1)] = 4$



Q.3 (2)

$x = 0,\ x = \frac{\pi}{a}$ are successive points of inflection



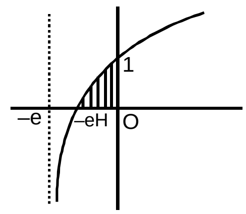
Graph of $y = \sin ax$

Area $= \int_0^{\pi/a} \sin ax \, dx = \frac{2}{a}$

Q.4 (1)

Area $= \int_0^{\pi/2} \cos x \, dx - \int_{\pi/2}^{\pi} \cos x \, dx = 2$

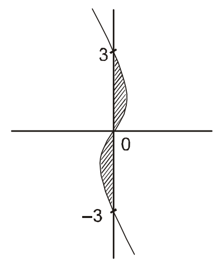
Q.5 (4)



$A = \int_0^1 (e^y - e)\, dy = 1$

Q.6 (3)

$x = 0 \Rightarrow y = 0, -3, 3$



Figure

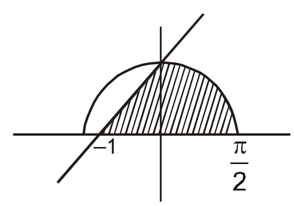
Required area $= 2\int_0^3 (9y - y^3)\, dy = \frac{81}{2}$

Q.7 (4)

Area $= 2\int_0^2 2\sqrt{x}\, dx = \frac{16\sqrt{2}}{3}$

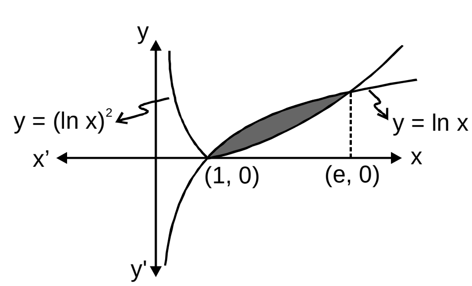
Q.8 (4)

From figure it is clear that required



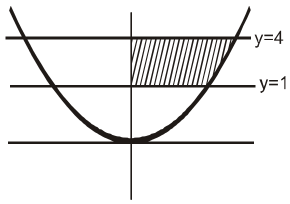
area $= \frac{1}{2} + \int_0^{\pi/2} \cos x \, dx = \frac{3}{2}$

Q.9 (3)



$A = \int_1^e (\ell n^2 x - \ell n\, x)\, dx = 3 - e$

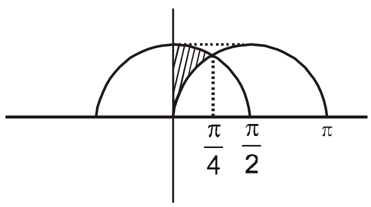
Q.10 (3)



Graph of $y = 4x^2$

Area = $\displaystyle\int_1^4 \frac{1}{2}\sqrt{y}\,dy = \frac{7}{3}$

Q.11 (3)



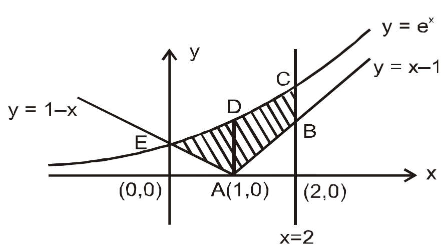
From figure

Area = $\displaystyle\int_0^{\pi/4}(\cos x - \sin x)dx = \sqrt{2} - 1$

Q.12 (3)

Area = $\displaystyle\int_0^1 \left(e^x - (1-x)\right)dx + \int_1^2 \left(e^x - (x-1)\right)dx$

$= \left(e^x - x + \frac{x^2}{2}\right)_0^1 + \left(e^x - \frac{x^2}{2} + x\right)_1^2$



Figure

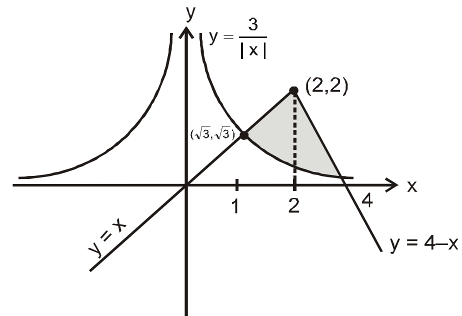
$= \left(e^1 - 1 + \frac{1}{2}\right) - 1 + \left(e^2 - 2 + 2\right) - \left(e^1 - \frac{1}{2} + 1\right)$

$= e^2 - 2$

Q.13 (2)

$A = \displaystyle\int_{\sqrt{3}}^{2}\left(x - \frac{3}{x}\right)dx + \int_2^3\left(4 - x - \frac{3}{x}\right)dx$

$= \left(\frac{x^2}{2} - 3\ell nx\right)_{\sqrt{3}}^{2} + \left(4x - \frac{x^2}{2} - 3\ell nx\right)_2^3$



Figure

$= \dfrac{4 - 3\ell n3}{2}$

Q.14 (1)

$\displaystyle\int_0^t (f(x) - (x^4 - 4x^2))dx = k\int_0^t (2x^2 - x^3 - f(x))dx$

Differentiating w.r.t. t

$f(t) = \dfrac{t^4 - kt^3 + (2k-4)t^2}{1+k}$

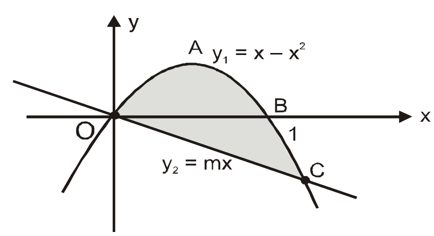
Q.15 (2)

$y = x - x^2$: $y = mx$

first find point of intersection : $x - x^2 = mx$

$x^2 + (m-1)x = 0 \Rightarrow x = 0, 1 - m$

Case - I



Figure

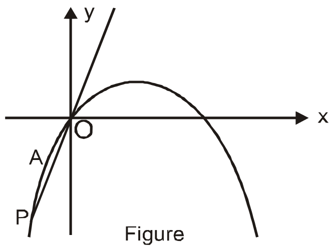
Area of OABCO

$A = \displaystyle\int_0^{1-m}(y_1 - y_2)dx$

$= \displaystyle\int_0^{1-m}(x - x^2 - mx)dx = 9/2$

= **m = – 2**

Case - II



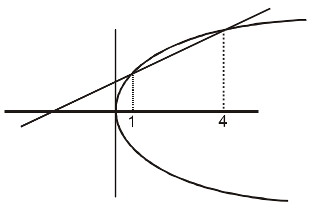
Area of PAOP

$$\int_{1-m}^{0}(x - x^2 - mx)\,dx = 9/2$$

$\Rightarrow$ m = 4

Q.16 (1)

Solving x = 1, 4



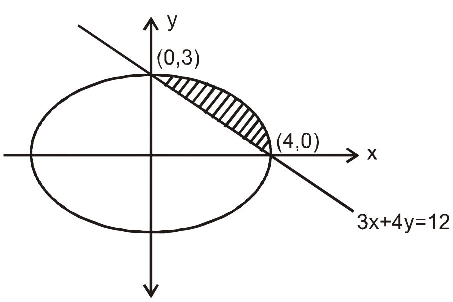
From graph it is clear that required

area $= \int_{1}^{4}\left(2\sqrt{x} - \frac{1}{3}(2x+4)\right)dx = \frac{1}{3}$

Q.17 (2)

Area $= \int_{0}^{4}\left(3\sqrt{1-\frac{x^2}{16}} - \frac{1}{4}(12-3x)\right)dx$

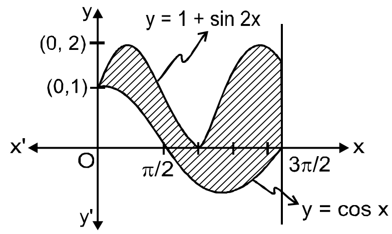
$$= \frac{3}{4}\left[\frac{x}{4}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4 - \frac{1}{4}\left[12x - \frac{3x^2}{2}\right]_0^4$$



$$= \frac{3}{4}[(0+4\pi)-(0+0)] - \frac{1}{4}[48-24]$$

$= 3\pi - 6 = 3(\pi - 2)$

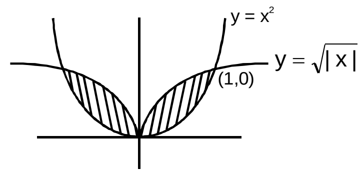
Q.18 (3)



$$A = \int_{0}^{3\pi/2}(1+\sin 2x - \cos x)dx$$

$$A = \int_{0}^{3\pi/2}(1+\sin 2x - \cos x)\,dx = 2 + \frac{3\pi}{2}$$

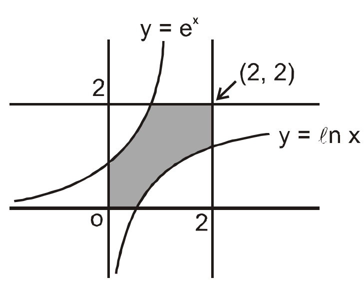
Q.19 (2)



$$A = 2\int_0^1\left(\sqrt{|x|} - x^2\right)dx = \frac{2}{3}$$

Q.20 (1)

$$A = \int_1^2 \ell n\, x\, dx = 2\,\ell n\, 2 - 1$$



$\Rightarrow$ Required area = 4 – 2 (ℓn 2 – 1) = 6 – 4 ℓn 2

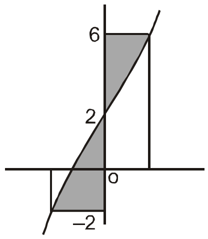
Q.21 (3)

The required area will be equal to the area enclosed by y = f(x),

y-axis between the abscissa at y =–2 and y = 6

Hence, Area

$$= \int_0^1 (6 - f(x))\,dx + \int_{-1}^{0}(f(x) - (-2))\,dx = \frac{9}{2}$$

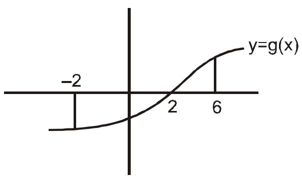


Graph of y = f(x)

Alternative

Clearly g(x) < 0 for x < 2 and g(x) > 0 for x > 2

$$\text{Area} = -\int_{-2}^{2} g(x)\,dx + \int_{2}^{6} g(x)\,dx$$



Figure

put x = f(t)

$$= -\int_{-1}^{0} t f'(t)\,dt + \int_{0}^{1} t f'(t)\,dt = \frac{9}{2}$$

Q.22 (1)

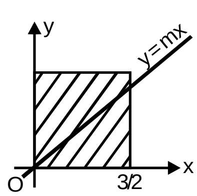
$\frac{d^2y}{dx^2} = 0$ at x = 2 so A

$$= \int_0^2 xe^{-x}\,dx = 1 - 3e^{-2}$$

Q.23 (1)

$$A = \int_0^{3/2} y\,dx = \frac{39}{8}$$

and $\left(\frac{39}{8}\right) \times \frac{1}{2} = \int_0^{3/2} mx\,dx$



$\Rightarrow$ m = 13/6

Q.24 (1)

$$\sin 2x - \sqrt{3}\sin x = 0 \Rightarrow \sin x\left(\cos x - \frac{\sqrt{3}}{2}\right) = 0$$

x = 0 on $\pi/6$

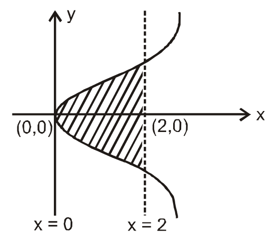
so $A = \int_0^a (\sin 2x - \sqrt{3}\sin x)\,dx$

$\Rightarrow$ 4A + 8 cos a = 7.

Q.25 (2)

$$\text{Area} = \int_1^3 \left(\frac{1}{x^2}\right) dx = \left(-\frac{1}{x}\right)_1^3 = -\frac{1}{3} - (-1) = \frac{2}{3}$$

Q.26 (3)



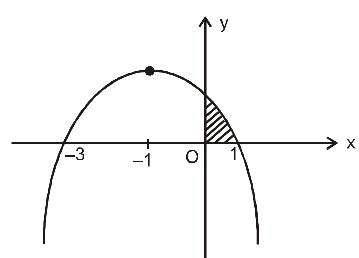
$$A = 2\int_0^2 x^3\,dx = 8$$

Q.27 (3)

$0 < y < 3 - 2x - x^2$, x > 0

$y = 3 - 2x - x^2 \Rightarrow (x + 1)^2 = -(y - 2)$

vertex (–1, 2) at y = 0, x = –3, x = 1



Figure

$$\text{Area} = \int_0^1 (3 - 2x - x^2)\,dx$$ **Ans.**

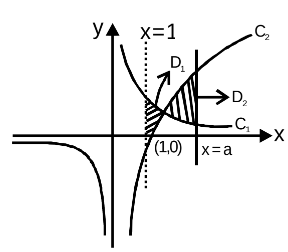
Q.28 (2)

$C_1 : y = \frac{1}{x} \Rightarrow xy - 1$

$C_2 : y = \ell n\,x$

x = 1, x = a

$D_1 = D_2$



$D_1 = D_2$

$$\int_1^c \left(\frac{1}{x} - \log x\right) dx = \int_c^a \left(\log x - \frac{1}{x}\right) dx$$

Q.29 (3)

$$\int_1^b f(x)\, dx = (b-1)\sin(3b+4)$$

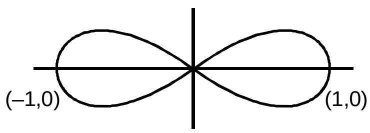
differentiate w.r.t. 'b'

$f(b)\cdot 1 = 3(b-1)\cos(3b+4) + \sin(3b+4)$

so $f(x) = 3(x-1)\cos(3x+4) + \sin(3x+4)$

Q.30 (2)

curve is symmetric about both the axes & cuts x-axis at (–1, 0) (0, 0) & (1, 0)

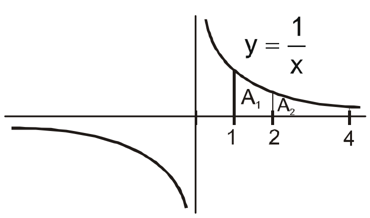


$$\text{Area of loop} = 2\int_0^1 x\sqrt{1-x^2}\, dx$$

$$= 2.\frac{2}{3} = \frac{4}{3}$$

JEE-ADVANCED
OBJECTIVE QUESTIONS

Q.1 (D)



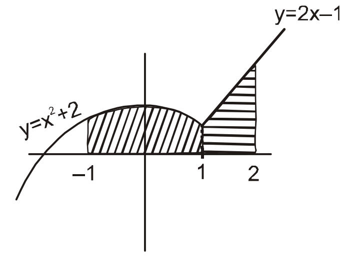
Figure

$$A_1 = \int_1^2 \frac{1}{x} dx = \ell n 2, \quad A_2 = \int_2^4 \frac{1}{x} dx = \ell n 2$$

$A_1 = A_2$

Q.2 (A)

$$\text{Area} = \int_{-1}^{1} (2 - x^2)\, dx + \int_1^2 (2x-1)\, dx = \frac{16}{3}$$

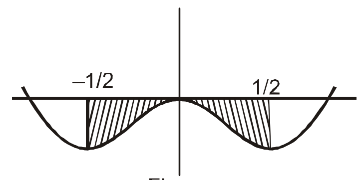


Figure

Q.3 **(D)**

$$\frac{dy}{dx} = 8x^3 - 2x. \quad \frac{dy}{dx} = 0 \Rightarrow (4x^2 - 1)\, x = 0$$

$$\Rightarrow x = -\frac{1}{2}, 0, \frac{1}{2}$$



Figure

$$\text{Required area} = -2\int_0^{1/2} (2x^4 - x^2)\, dx = \frac{7}{120}$$

Q.4 (A)

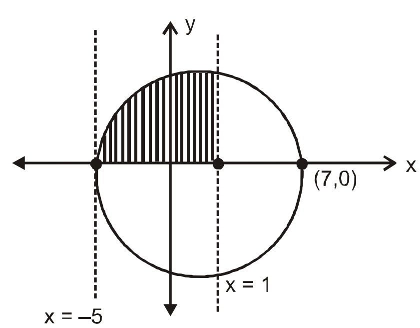
$y^2 = (7-x)(5+x)$ at $y = 0$, $x = 7, -5$

$(x-1)^2 + y^2 = (6)^2$ centre (1,0)

From the figure it is clear that area is

$$\int_{-5}^{1} y\, dx = \int_{-5}^{1} \sqrt{(6)^2 - (x-1)^2}\, dx$$

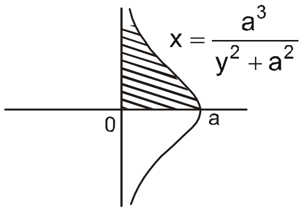
$$= \left[\frac{x-1}{2}\sqrt{6^2 - (x-1)^2} + \frac{36}{2}\sin^{-1}\left(\frac{x-1}{6}\right)\right]_{-5}^{1}$$



$$= \left|(0+0) - (0 + 18\sin^{-1}(-1))\right| = 9\pi$$

Q.5 (C)

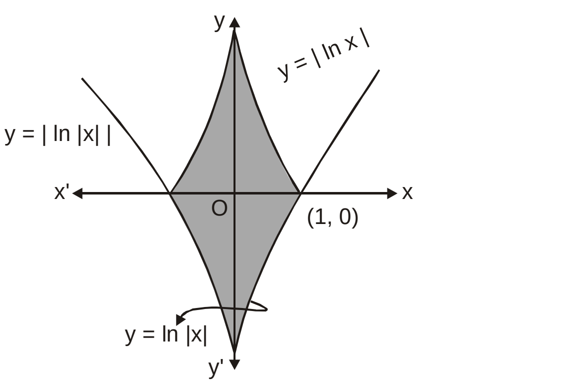
x = 0 is asymptote



figure

Required area = $2\int_0^{\infty} \frac{a^3}{y^2 + a^2} dy = a^2$

Q.6 (B)



Area enclosed by the curves y = ln x, y = ln |x|

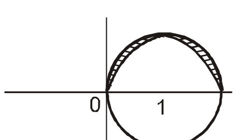
y = |ln x| and y= | ln |x| | is $4\int_0^1 |\ln x|\, dx$

$= 4[x \ln x - x]_0^1 = 4$

Q.7 (A)

$(x - 1)^2 + y^2 = 1$

Area of circle is πr^2



Figure

Required area = $\frac{\pi}{2} - \int_0^2 \sin\frac{\pi x}{2} dx$

$= \frac{\pi}{2} - \frac{4}{\pi}$

Q.8 (A)

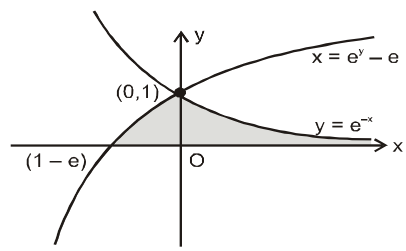
$y = \log_e(x + e)$, at y = 0, x = 1 – e point (1 – e, 0)

$e^y = x + e \Rightarrow e^y - e = x$

$x = \log_e\left(\frac{1}{y}\right)$, at y = e , x = –1

$e^x = \frac{1}{y} \Rightarrow y = e^{-x}$

Area = $\int_{1-e}^{0} \log_e(x+e)dx + \int_0^{\infty} e^{-x} dx$



Figure

$= \int_1^e \ell nt\, dt + \int_0^{\infty} e^{-x} dx = (t\ell nt - t)_1^e + \left(\frac{e^{-x}}{-1}\right)_0^{\infty}$

= [(e – e) – (0 –1)] + $[-e^{-\infty} + 1]$ = 1 + 1 = 2

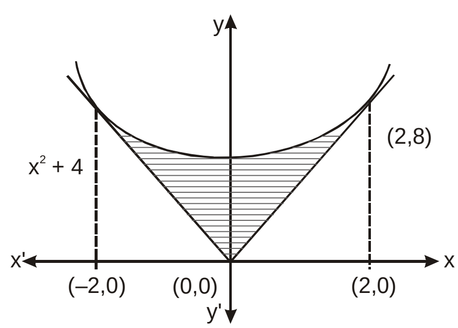
Q.9 (B)

h(x) = f{g(x)} $\Rightarrow$ h(x) = f(2x + 1)

$4x^2 + 4x + 5 = f(2x + 1)$

$\Rightarrow f(2x + 1) = (2x + 1)^2 + 4$

$f(t) = l^2 + 4 \Rightarrow f(x) = x^2 + 4$



Let the pair of S.l assing through the origin is

y = mx and tangent to $y = x^2 + 4$

$\therefore x^2 + 4 = mx \Rightarrow x^2 = mx + 4 = 0 \Rightarrow D = 0$

$\Rightarrow m^2 - 16 = 0 \quad \Rightarrow m = \pm 4 \quad \Rightarrow \; x \tan y = \pm 4x$

Point of untense of $y = 4x$ and $y = x^2 + 4$

$x^2 + 4 = 4x \Rightarrow (x-2)^2 = 0 \Rightarrow x = 2$

$$A = 2\int_0^2 (x^2+4)dx - 2 \times \frac{1}{2} \times 2 \times 8 \Rightarrow A = \frac{16}{3}$$

Q.10 (D)

$$\frac{dy}{dx} = 1 + \cos x \geq 0, \; \forall x$$

$$\frac{d^2y}{dx^2} = -\sin x$$

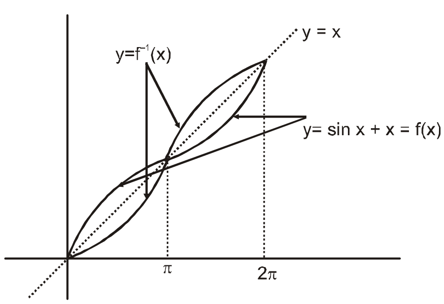
$x = n\pi$ are points of inflection, where curve changes its concavity.

For $x \in (0, \pi)$, $\sin x > 0 \Rightarrow x + \sin x > x$.

For $x \in (\pi, 2\pi)$, $\sin x < 0 \Rightarrow x + \sin x < x$.

Required area = 4A

$$A = \int_0^{\pi} ((x + \sin x) - x)dx$$



Figure

A = 2

Required area = 4A = 8

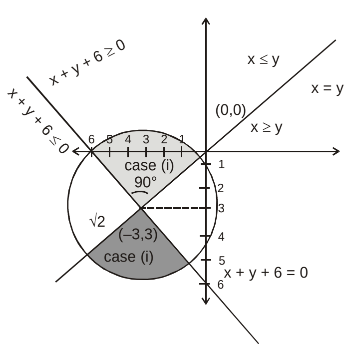
Q.11 (C)

$f(x) = x^2 + 6x + 1 \Rightarrow f(y) = y^2 + 6y + 1$

$f(x) + f(y) = x^2 + y^2 + 6x + 6y^{+2} \leq 0$

snous interior area of circle radius of circle is = 4

$f(x) - f(y) = x^2 - y^2 + 6x - 6y \leq 0$



$\Rightarrow (x-y)(x+y+6) \leq 0$

Case (i) $x - y \geq 0$ and $x + y + 6 \leq 0$

or Case (ii) $x - y \leq 0$ and $x^2 + y + 6 \geq 0$,

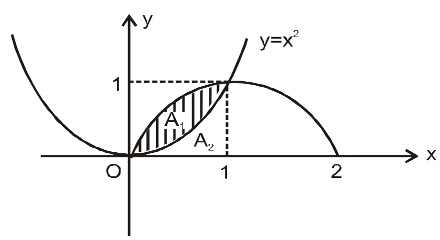
$$A = \frac{1}{2} \times 16 \times \frac{\pi}{2} \times 2 \Rightarrow A = 8\pi$$

Q.12 (D)

$$A_1 = \int_0^1 \left(\sin\frac{\pi}{2}x - x^2\right)dx = \left(\frac{-\cos\frac{\pi}{2}x}{\pi/2} - \frac{x^3}{3}\right)_0^1$$

$$= \frac{2}{\pi} - \frac{1}{3} = \frac{6-\pi}{3\pi}$$

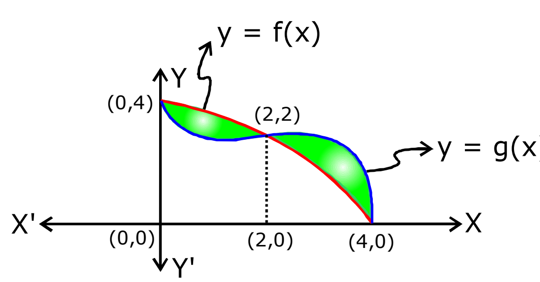
$$A_2 = \int_0^1 x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$$



Figure

$$\therefore \frac{A_1}{A_2} = \frac{6\pi/3\pi}{1/3} = \frac{6-\pi}{\pi}$$

Q.13 (C)

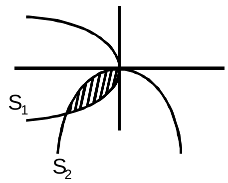


$$\int_0^4 (f(x) - g(x))dx = \int_0^2 (f(x) - g(x))dx + \int_2^4 (g(x) - f(x))dx$$

$$\int_0^2 (f(x)\, g(x)\, dx = 15$$

Q.14 (B)

Area of shaded region = $\frac{1}{3}$



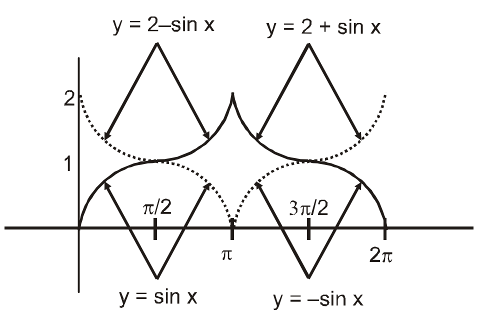
Q.15 (B)

$f(x) + f(\pi - x) = 2. \quad \forall \; x \in \left(\frac{\pi}{2}, \pi\right]$

$f(x) = 2 - \sin(\pi - x)$

$f(x) = 2 - \sin x, \; \forall \; x \in \left(\frac{\pi}{2}, \pi\right]$

$f(x) = 2 - f(2\pi - x), \; \forall \; x \in \left(\pi, \frac{3\pi}{2}\right]$



$f(x) = 2 + \sin x, \; x \in \left(\pi, \frac{3\pi}{2}\right]$

$f(x) = f(2\pi - x), \; \forall \; x \in \left(\frac{3\pi}{2}, 2\pi\right]$

$f(x) = -\sin x, \; \forall \; x \in \left(\frac{3\pi}{2}, 2\pi\right]$

Clearly, from figure required area = 2π

Q.16 (A)

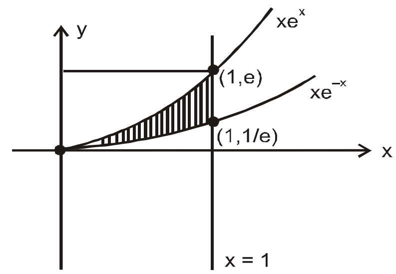
$y = xe^{x}, \; y = xe^{-x}$

$xe^{x} = x^{-x} \Rightarrow x(e^{x} - e^{-x}) = 0$

$x = 0$, Intersection point (0,0)

$$\text{Area} = \int_0^1 (xe^{x} - xe^{-x})\,dx$$

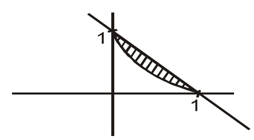
$$= x\left(e^{x} + e^{-x}\right)_0^1 - \int_0^1 1.(e^{x} + e^{-x})\,dx$$



Figure

$$= \left(e + \frac{1}{e} - 0\right) - \left[e^{x} - e^{-x}\right]_0^1 = \frac{2}{e}$$

Q.17 (A)



Figure

$$\text{Area} = \int_0^1 \left((1 - x) - (1 - \sqrt{x})^2\right)dx$$

$$= \frac{4}{3} - 1 = \frac{1}{3}$$

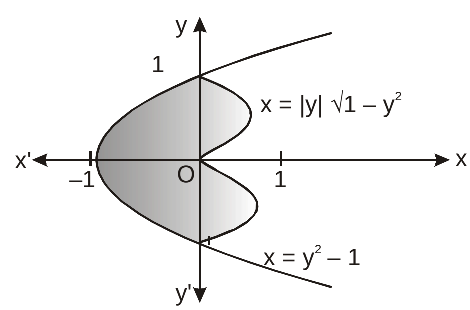
Q.18 (B)

$$A = \int_{1/2}^{e} (x(1 - \ell n\, x))\,dx = \frac{e^2 - 5e^{-2}}{4}$$

Q.19 (D)

$$A = 5 - \int_0^1 (3x^3 + 2x)\,dx = \frac{13}{4}$$

Q.20 (D)



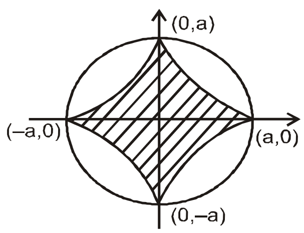
Area = $2\int_0^1 y\sqrt{1-y^2}\,dy + 2\int_0^1 (y^2-1)dy \Rightarrow$ A = 2 **Ans.**

Q.21 (A)

$x = a\cos^3 t,\ \ y = a\sin^3 t \ \Rightarrow\ x^{2/3} + y^{2/3} = a^{2/3}$

$$A = 4\int_0^{\pi/2} y\frac{dx}{dt}dt = 4\int_0^{\pi/2} 3a^2\sin^3 t\cos^2 t(-\sin t)\,dt$$

$$= \left|-12a^2\int_0^{\pi/2}\sin^4 t\cos^2 t\,dt\right| = \left|-12a^2\,\frac{3.1.1}{6.4.2.1}\times\frac{\pi}{2}\right|$$



Figure

$= \frac{3}{8}\pi a^2$ sq. units

Q.22 (C)

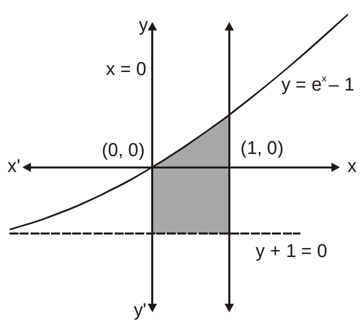
According to questions f''(x) = f'(x)

$\Rightarrow \int f''(x)dx = \int f'(x)dx$

f'(x) = f(x) + 4 ⇒ f'(0) = f(0) + 4 ⇒ 4 = 1

Now f'(x) = f(x) + 1

$\Rightarrow \frac{f'(x)}{f(x)+1} = 1 \Rightarrow \int\frac{f'(x)}{f(x)+1}dx = \int dx$



$\ln|f(x)+1| = x + C_2 \Rightarrow \ln|f(0)+1| = 0 + C_2 \Rightarrow C_2 =$

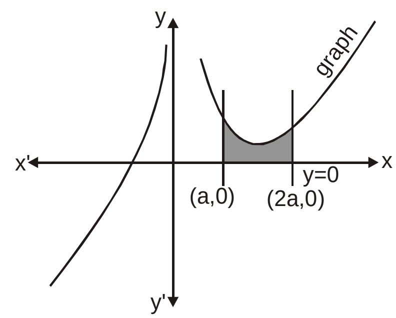
$\ln|f(x)+1 = x \Rightarrow f(x) + 1 = e^x \Rightarrow f(x) = e^x - 1$

$A = \int_0^1 (e^x - 1)dx + 1\times 1 \Rightarrow$ A = e – 2 + 1 ⇒ A = e – 1

Q.23 (D)

$$A = \int_a^{2a}\left(\frac{x}{6}+\frac{1}{x^2}\right)dx$$

$$A = \left[\frac{x^2}{12} + -\frac{1}{x}\right]_a^{2a}$$



$$A = \frac{a^2}{4} + \frac{1}{2a}$$

Now, $\frac{dA}{da} = \frac{a}{2} - \frac{1}{2a^2}$

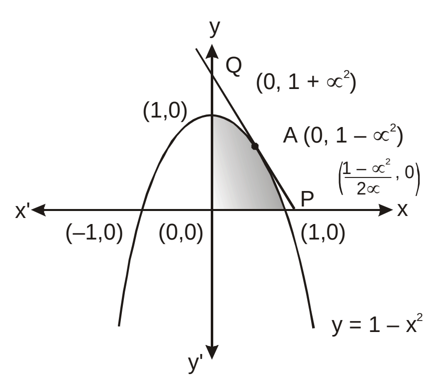
For maximum and minimum value of A

$\frac{dA}{da} = 0 \Rightarrow \frac{a}{2} = \frac{1}{2a^2} \Rightarrow a^3 = 1$

$\therefore \frac{d^2A}{da^2} = \frac{1}{2} + \frac{1}{a^3}$

$\left.\frac{d^2A}{da^2}\right)_{a=1} = \frac{1}{2} + 1 > 0 \Rightarrow$ A is least value at a = 1

Q.24 (A)



Example of tangent at A($\propto$, 1 – $\propto^2$) for the curve $y = 1 - x^2$ is $y + 1 - \propto^2 = 2 - 2\propto x$

$2 \propto x + y = 1 + \propto^2 \Rightarrow \frac{x}{\frac{1+\propto^2}{2\propto}} + \frac{y}{1+\propto^2} = 1$

Area of $\Delta OPQ = \frac{1}{2} \times \frac{(1+\propto^2)^2}{2\propto} \Rightarrow \Delta = \frac{(1+\propto^2)^2}{4\propto}$

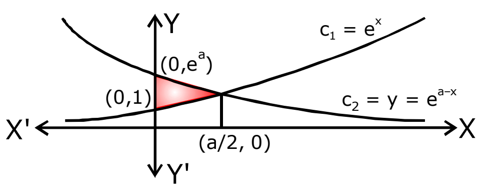
$\frac{d\Delta}{d\propto} = \frac{(\propto^2+1)(3\propto^2-1)}{\propto}$ for minimum value of Δ

$\frac{d\Delta}{d\propto} = 0 \Rightarrow \frac{(3\propto^2-1)(\propto^2+1)}{\propto} = 0 \Rightarrow \propto = \frac{1}{\sqrt{3}}$

Minimum Area of $\Delta = \frac{4\sqrt{3}}{9}$

According to questions $\Rightarrow \sqrt{4+12} = 4$

Q.25 (D)

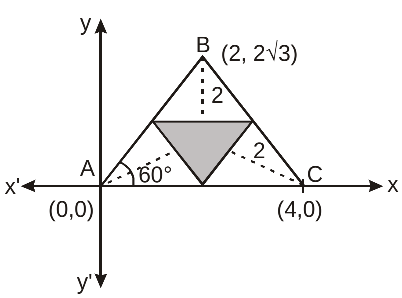


$S = \int_0^{a/2} (e^a e^{-x} - e^x)\, dx$

& $\lim_{a\to0} \frac{s}{a^2} = \lim_{a\to0} \frac{e^a - 2e^{a/2} + 1}{a^2} = \frac{1}{4}$

Q.26 (B)

Area bounded by the curve traced by P is



$= \frac{\sqrt{3}}{4} \times (4)^2 - 3 \times \frac{1}{2} \times 4 \times \frac{\pi}{3}$

$= 4\sqrt{3} - 2$ sq. unit.

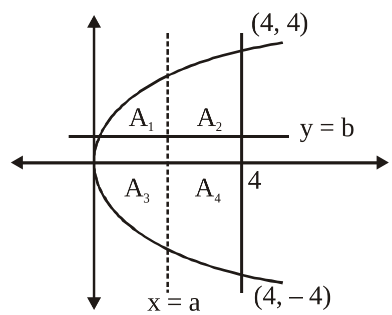
JEE-ADVANCED
MCQ/COMPREHENSION/COLUMN MATCHING

Q.1 (A,B,C)

$A_1 = A_2 = A_3 = A_4 = \frac{1}{4} \cdot \frac{2}{3} \cdot 32 = \frac{16}{3}$

$A_2 = \int_a^4 \left(2\sqrt{x} - b\right) dx = \frac{16}{3}$

$A_4 = \int_a^4 \left(b + 2\sqrt{x}\right) dx = \frac{16}{3} \Rightarrow b = 0$

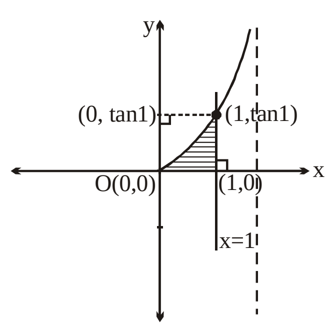


$A_4 = \int_a^4 2\sqrt{x}\, dx = \frac{16}{3}$

$\frac{4}{3} x^{\frac{3}{2}} \Big|_a^4 = \frac{16}{3} \Rightarrow \frac{32}{3} - \frac{4}{3} a^{\frac{3}{2}} = \frac{16}{3}$

$\frac{4}{3} a^{\frac{3}{2}} = \frac{16}{3} \Rightarrow a^3 = 16$]

Q.2 (A,D)



Required area $= \int_0^1 \tan x\, dx = \int_0^1 \tan(1-x)\, dx$

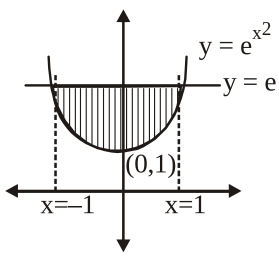
(Using king property)

Also, required area $= \tan 1 - \int_0^{\tan 1} (\tan^{-1} y)\, dy$.

Q.3 (A,B,C,D)

$$A = \int_{-1}^{1} \left(e - e^{x^2}\right) dx \quad = 2e - \int_{-1}^{1} e^{x^2}\, dx$$

$$A = 2\int_1^e x\, dy = 2\int_1^e \sqrt{\ln y}\, dy$$



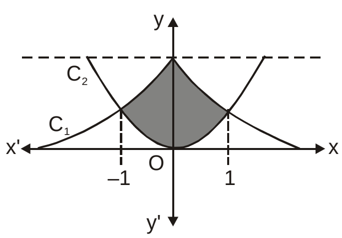
put $\ln y = t$; $y = e^t$; $dy = e^t\, dt$

$$A = 2\int_0^1 \sqrt{t}\, e^t dt .$$

Q.4 (A,B)

Point of intersection of two curves C_1 and C_2

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$



$x^4 + x^2 - 2 = 0$

$(x^2 + 2)(x^2 - 1) = 0$

$x = \pm 1$

Area bounded by the curve C_1 and $y = 0$ is

$$= 2\int_0^\infty \frac{1}{1+x^2} dx = 2\left[\tan^{-1} x\right]_0^\infty = 2\left[\frac{\pi}{2}\right] = \pi$$

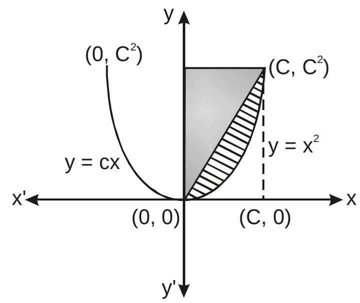
Area bounded by

C_1 and C_2 is $2\left(\int_0^1 \frac{1}{1+x^2} dx - \int_0^1 \frac{x^2}{2} dx\right)$

$$= 2\left[\tan^{-1} x\right]_0^1 - \frac{1}{3}\left[x^3\right]_0^1 = \frac{\pi}{3} - \frac{1}{3}$$ **Ans.**

Q.5 (A,C)

$$\text{Area (T)} = \frac{1}{2} \times C^2 \times C = \frac{C^3}{2}$$



$$\text{Area (R)} = \int_0^C (Cx - x^2)dx = \frac{C^3}{6}$$

$$\lim_{C\to 0^+} \frac{\text{Area(T)}}{\text{Area(R)}} = \lim_{C\to 0^+} \frac{6C^3}{2\times C^3} = 3$$

Q.6 (A, B, C,D)

$h'(x) = 3x^2 - 4x + c$

$h'(1) = 0 \Rightarrow c = 1$

$h'(x) = 3x^2 - 4x + 1$

$h(x) = x^3 - 2x^2 + x + d$

$5 = 1 - 2 + 1 + d \Rightarrow d = 5$

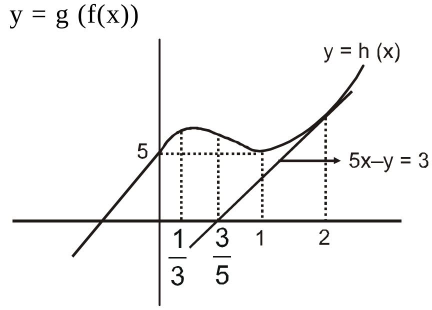
$h(x) = x^3 - 2x^2 + x + 5$

$h'(2) = 3\times 2^2 - 4\times 2 + 1 = 5.$

Tangent at (2, 7) is $y - 7 = 5(x - 2)$

$5x - y = 3$

$y = g(f(x))$



Figure

$y = 0$

$h'(x) = 0 \Rightarrow x = \frac{1}{3}, 1$

Area by $y = h(x)$, $y = g(f(x))$, $x = 0$, $x = 2$ is

$$\int_0^2 \left(x^3 - 2x^2 + x + 5\right)dx = \frac{32}{3} .$$

Area by ordinates of local maxima $\left(x = \frac{1}{3}\right)$, local minima $(x = 1)$, $y = h(x)$ and x - axis is

$$\int_{1/3}^{1}(x^3-2x^2+x+5)\,dx$$

$$= \int_{\frac{1}{3}}^{1}\left(\left(1+\frac{1}{3}-x\right)^3-2\left(1+\frac{1}{3}-x\right)^2+\left(1+\frac{1}{3}-x\right)+5\right)dx.$$

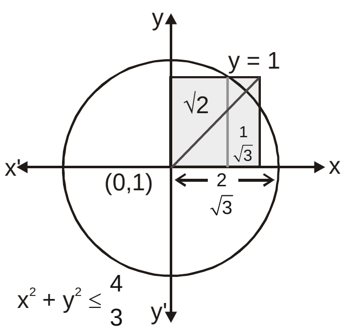
Area bounded by y = h (x), y = 5x – 3 and x = 0 is

$$\int_{0}^{2}\left((x^3-2x^2+x+5)-(5x-3)\right)dx = \frac{20}{3}$$

Q.7 (A,C)

$$\frac{2}{\sqrt{3}} = \frac{2}{1.732} = 1.15$$

$$x^2 + 1 = \frac{4}{3}$$



$$x^2 = \frac{1}{3} \Rightarrow x = \frac{1}{\sqrt{3}}$$

$$R_1\pi R_2 = \frac{1}{\sqrt{3}} \times 1 + \int_{\frac{1}{\sqrt{3}}}^{1}\sqrt{\frac{4}{3}-x^2}\,dx$$

$$R_1\pi R_2 = \frac{1}{\sqrt{3}} + \left[\frac{x}{2}\sqrt{\frac{4}{3}-x^2}+\frac{2}{3}\sin^{-1}\left(\frac{x\sqrt{3}}{2}\right)\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$R_1\pi R_2 = \frac{1}{\sqrt{3}} + \left[\frac{x}{2}\sqrt{\frac{4}{3}-x^2}+\frac{2}{3}\sin^{-1}\left(\frac{x\sqrt{3}}{2}\right)\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= \frac{1}{2\sqrt{3}} + \frac{2\pi}{9} - \frac{\pi}{9} = \frac{1}{2\sqrt{3}} + \frac{\pi}{6} = \frac{1}{\sqrt{3}} + \frac{\pi}{9}$$

$$= \frac{3\sqrt{3}+\pi}{9}$$

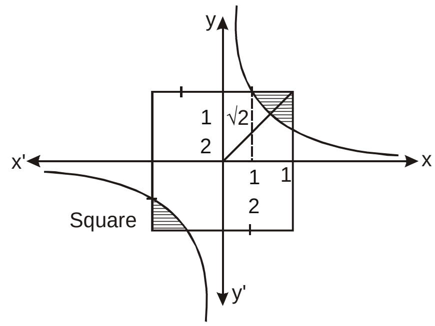
Q.8 (A,D)

$|x| \le 1, |y| \le 1$

$\Rightarrow -1 \le x \le 1$

& $1 \le y \le 1$

& $xy \le \frac{1}{2} \quad \Rightarrow y \le \frac{1}{2x}$



Required Area = $2\int_{\frac{1}{2}}^{1}\left(1-\frac{1}{2x}\right)dx$

$= 2\left(x-\frac{1}{2}\tan x\right)_{\frac{1}{2}}^{1}$ = 1 – ln 2 sq. unit

Comprehension # 1 (Q. No. 9 to 11)

Q.9 (A)

Q.10 (C)

Q.11 (B)

$2f''(x)\, f'(x) = 2\, f(x)\, f'(x)$

Integrating

$(f'(x))^2 = (f(x))^2 + c$

put x = 0 ⇒ c = 5

$(f'(x))^2 = (f(x))^2 + 5$

put y = f(x)

$$\frac{dy}{dx} = \pm\sqrt{y^2+5}$$

$$\ell n\left(y+\sqrt{y^2+5}\right) = \pm x + c_1$$

$x = 0, y = 2 \Rightarrow c_1 = \ell n 5$

$$\frac{y+\sqrt{y^2+5}}{5} = e^{\pm x}$$

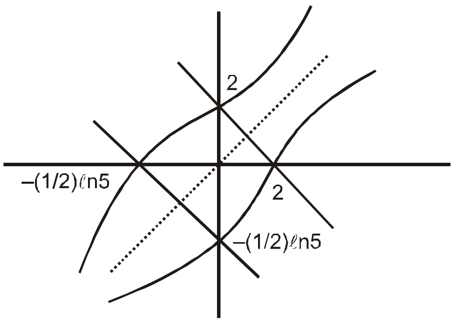
$$y = \frac{5e^{x}-e^{-x}}{2} \text{ or } y = \frac{5e^{-x}-e^{x}}{2}$$

If $f(x) = \frac{5e^{-x}-e^{x}}{2}$; $f'(0) = 3$ is not satisfied

$\Rightarrow f(x) = \dfrac{5e^x - e^{-x}}{2}$

put $f(x) = 0 \Rightarrow 2x = \ell n\left(\dfrac{1}{5}\right) \Rightarrow x = -\dfrac{1}{2}\ell n5$

$f'(x) = \dfrac{5e^x + e^{-x}}{2} > 0 \Rightarrow f(x)$ is increasing



Figure

Area in second quadrant = $\displaystyle\int_{-\frac{1}{2}\ell n5}^{0}\left(\frac{5e^x - e^{-x}}{2}\right)dx$

$= \left.\dfrac{5e^x + e^{-x}}{2}\right]_{-\frac{1}{2}\ell n5}^{0} = 3 - \sqrt{5}$

Area by lines $x + y = 2$, $x + y = -\dfrac{1}{2}\ell n5$,

$y = f(x)$ and $y = f'(x)$ is $2(3 - \sqrt{5}) + \dfrac{1}{2}.\ 2.2 + \dfrac{1}{2}\left(\dfrac{1}{2}\ell n5\right)\left(\dfrac{1}{2}\ell n5\right)$

$= 8 - 2\sqrt{5} + \dfrac{1}{8}(\ell n5)^2$

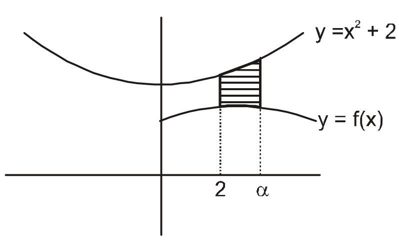
Comprehension # 2 (Q. No. 12 to 14)

Q.12 (B)

Q.13 (C)

Q.14 (D)

(12, 13 & 14)

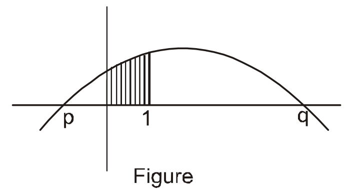


Figure

$\alpha^3 - 4\alpha^2 + 8 = \displaystyle\int_2^{\alpha}(x^2 + 2 - f(x))dx$

$3\alpha^2 - 8\alpha = \alpha^2 + 2 - f(\alpha)$

$f(x) = -2x^2 + 8x + 2$

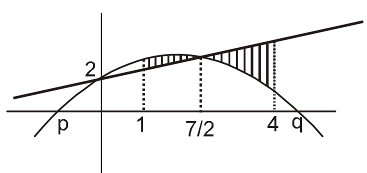


Figure

Area = $\displaystyle\int_0^1(-2x^2 + 8x + 2)dx = \frac{16}{3}$

Sum of roots = p + q = 4

Area = $\displaystyle\int_1^{7/2}(-2x^2 + 8x + 2 - x - 2)dx + \int_{7/2}^{4}(x + 2 + 2x^2 - 8x - 2)dx$



Figure

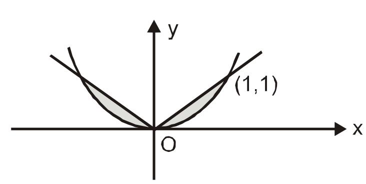
$= \dfrac{79}{12}$

Q.15 (A) → (q), (B) → (s), (C) → (p), (D) → (r)

(A) $\displaystyle\int_1^2 x^3dx = \left(\frac{x^4}{4}\right)_1^2 = \frac{15}{4}$

(B) $\displaystyle\int_0^4(4x - x^2)dx = \left(2x^2 - \frac{x^3}{3}\right)_0^4 = \frac{32}{3}$

(C)



Figure

$$A = 2\int_0^1 (x - x^2)\,dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

(D) $A = \int_0^1 (x^2 - x^3)\,dx + \int_1^2 (x^3 - x^2)\,dx$

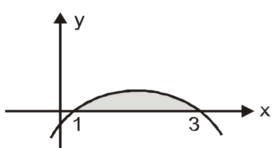
$$= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1 + \left(\frac{x^4}{4} - \frac{x^3}{3}\right)_1^2 = \frac{3}{2}$$

Q.16 (A) $\rightarrow$ (s), (B) $\rightarrow$ (s), (C) $\rightarrow$ (q), (D) $\rightarrow$ (p)

(A) $0 \le y \le 4x - x^2 - 3$

at $y = 0$, $x^2 - 4x + 3 = 0 \Rightarrow x = 1, x = 3$

$$A = \int_1^3 (4x - x^2 - 3)\,dx = \frac{4}{3}$$

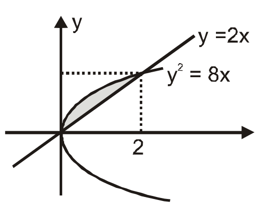


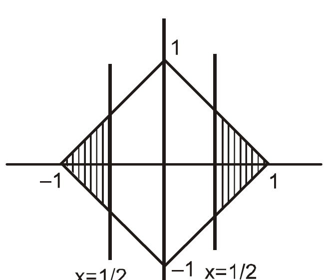
Figure

(B) $8x = 4x^2$

intersection point (0,0), (2,4)

$$A = \int_0^4 \left(\frac{y}{2} - \frac{y^2}{8}\right)dy = \left(\frac{y^2}{4} - \frac{y^3}{24}\right)_0^4 = 4 - \frac{8}{3} = \frac{4}{3}$$

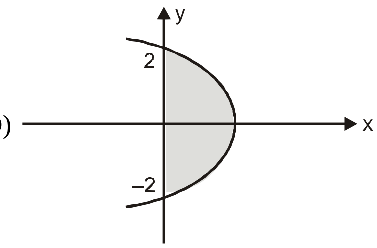


(C) 

Figure

Required area =

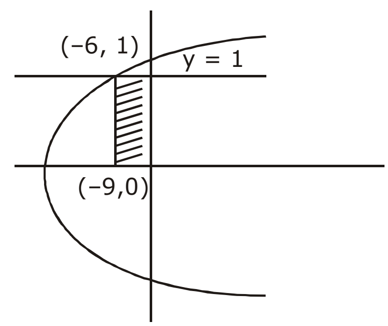
$$4\left(\frac{1}{2}.\frac{1}{2}.\frac{1}{2}\right) = \frac{1}{2}$$

(D) 

$$A = \int_{-2}^{2} (4 - y^2)\,dy = \left(4y - \frac{y^3}{3}\right)_{-2}^{2} = \frac{32}{3}$$

Q.17 A $\rightarrow$ R; B $\rightarrow$ S; C $\rightarrow$ P; D $\rightarrow$ Q

(A) $y^2 = \frac{1}{3}(x + 9)$



$\left|\int_0^{+1} (3y^2 - 9 + b)dy\right|$ + |area of rectangle|

2 + 6 = 8, square unit

(B) $f(x) = a\sqrt{x} + bx$

$\int_0^4 a\sqrt{x} + b(x)dx$

$\left|\frac{ax^{3/2}}{3/2} + \frac{bx^2}{2}\right|_0^4 = 8$

16a + 24b = 24 ...(1)

a + b = 2 ...(2)

from (1) & (2)

b = – 1, a = 3

Then value 2a + b = 6 – 1 = 5

(C) $\left|\int_0^{n/4} \cos^2 x - \sin^2 x\,dx\right|$

$+ \left|\int_{n/4}^{3n/4} \sin^2 x - \cos^2 x\,dx\right|$

$+\left|\int_{3\pi/4}^{\pi} \cos^2 x - \sin^2 x\,dx\right| = 1$

(D) $\int_0^{16/m} 4\sqrt{x} - mx\,dx$

$$\left[\frac{4x^{3/2}}{3/2} - \frac{mx^2}{2}\right]_0^{16/m} = \frac{2}{3}$$

m = 4

NUMERICAL VALUE BASED

Q.1. [1]

$y = mx + 2 \Rightarrow x = \left(\frac{y-2}{m}\right)$ ....(1)

$x = 2y - y^2$ ....(2)

$(y-1)^2 = -(x-1)$ vertex (1, 1)

From (1) and (2) $\frac{y-2}{m} = 2y - y^2$

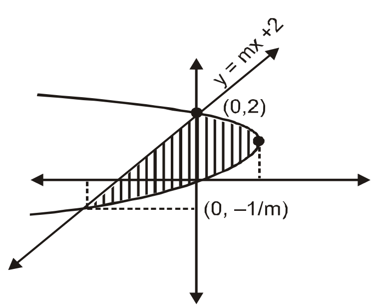
$\Rightarrow my^2 + (1-2m)y - 2 = 0 \quad \alpha\beta = -\frac{2}{m}$

$\alpha = 2, \quad \beta = -\frac{1}{m}$

$$\text{Area} = \int_{-1/m}^{2}\left[\left(2y - y^2\right) - \frac{(y-2)}{m}\right]dy$$

$$\frac{9}{2} = \left[\frac{2y^2}{2} - \frac{y^3}{3} - \frac{1}{m}\frac{y^2}{2} + \frac{2y}{m}\right]_{-1/m}^{2}$$

$$\Rightarrow \frac{9}{2} = \left(\frac{4}{3} + \frac{2}{m} + \frac{1}{6m^3} + \frac{1}{m^2}\right)$$



Figure

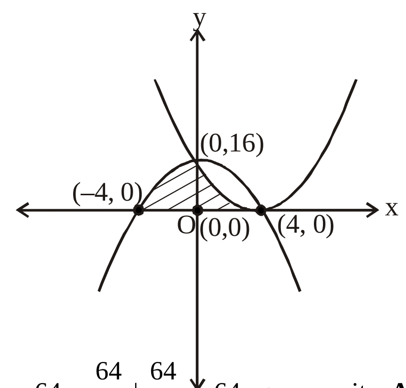
m = 1 satisfy the equation $\Rightarrow$ m = 1

Q.2 [64]

Clearly required area

$$= \int_{-4}^{0}\left(16 - x^2\right)dx + \int_0^4 (x-4)^2\,dx$$

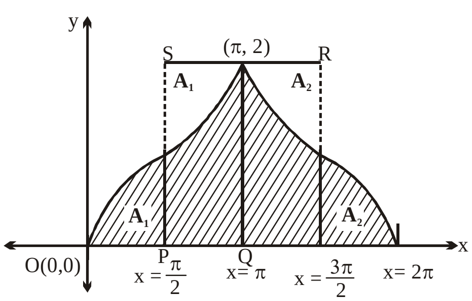
$$= \left(16x - \frac{x^3}{3}\right)_{-4}^{0} + \left.\frac{(x-4)^3}{3}\right|_0^4$$



$= 64 - \frac{64}{3} + \frac{64}{3} = 64$ square units. **Ans.**

Q.3 [4]

Area of rectangle PQRS $= \left(\frac{3\pi}{2} - \frac{\pi}{2}\right) \times 2 = 2\pi$



$= a\pi + b$ (Given)

So, a = 2, b = 0.

Hence $(a^2 + b^2) = 4$.

Alternatively :

As graph of f(x) is symmetrical with respective to the line $x = \pi$, so required area = 2A, where

$$A = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^{\pi} f(x)\,dx$$

$$= \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^{\pi} \left(2 - f(\pi - x)\right)dx$$

$$\Rightarrow A = 1 + 2\left(\frac{\pi}{2}\right) - \int_{\pi/2}^{\pi} f(\pi - x)\,dx$$

Put, $\pi - x = t$ and $2\pi - x = t$, so

$$A = 1 + \pi + \int_{\frac{\pi}{2}}^{0} f(t)\,dt = 1 + \pi - 1 = \pi$$

So, required area = 2A = $2\pi \equiv a\pi + b$ (Given) $\Rightarrow$ a = 2, b = 0

Hence $(a^2 + b^2) = 2$.

Q.4 [9]

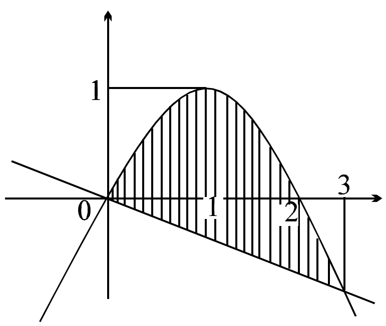
$y = 2x - x^2$

$$\frac{dy}{dx} = 2 - 2x = 0 \Rightarrow x = 1$$

solving $y = 2x - x^2$ and $y = -x$, we have

$-x = 2x - x^2 \Rightarrow x^2 - 3x = 0 \Rightarrow x = 0$ or 3

$$A = \int_0^3 \left[\left(2x - x^2\right) - (-x)\right] dx = \int_0^3 \left(3x - x^2\right) dx$$



$$= \left.\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2} = A \Rightarrow 2A = 9 \text{ sq. unit}$$

Q.5 [4] sq. units

$y = 1 + x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 2x$

Hence $m = -\frac{1}{2x_1}$ (m = slope of normal at x_1, y_1)

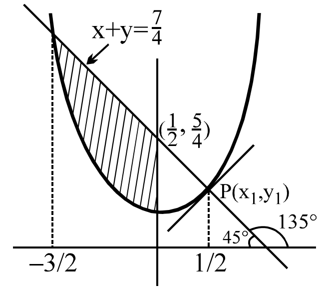
$\therefore \quad x_1 = \frac{-1}{2m} = \frac{1}{2}$; $y_1 = 1 + \frac{1}{4m^2} = \frac{5}{4}$

equation of normal $y - \left(1 + \frac{1}{4m^2}\right) = m\left(x + \frac{1}{2m}\right)$

put $m = -1$, we get $\quad y - \frac{5}{4} = -\left(x - \frac{1}{2}\right)$

$$x + y = \frac{5}{4} + \frac{1}{2} = \frac{7}{4} \quad \text{....(1)}$$

$$A = \int_{x_1}^{x_2} \left(\frac{7}{4} - x\right) - (1 + x^2)\,dx = \int_{x_1}^{x_2} \left(\frac{3}{4} - x - x^2\right) dx$$



now $\quad 1 + x^2 = -x + \frac{7}{4}$

$4x^2 + 4x - 3 = 0 \Rightarrow x = -\frac{3}{2}$ and $x = \frac{1}{2}$

$$\therefore \quad A = \int_{-3/2}^{1/2} \left(\frac{3}{4} - x - x^2\right) dx$$

simplifying $A = \frac{4}{3}$ sq. units = A $\Rightarrow$ 3A = 4 Ans.

Q.6 [19]

Let $\quad z = x + iy$

$\text{Re}(z + 1) = |z - 1|$

$(x + 1) = \sqrt{(x-1)^2 + y^2}$

$(x + 1)^2 = (x - 1)^2 + y^2 \quad \Rightarrow \quad y^2 = 4x$ ....(1)

again $\quad \text{Re}\left((1+i)(x+iy)\right) = 1$

$x - y = 1 \quad \Rightarrow \quad y = x - 1$ ....(2)

$$\text{Area} = \int_{y_1}^{y_2} \left((y+1) - \frac{y^2}{4}\right) dy \quad \text{.....(1)}$$

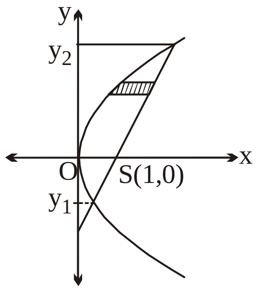
where y_1 and y_2 are the roots of the equation

$y^2 = 4(1 + y) \Rightarrow y^2 - 4y - 4 = 0 \Rightarrow y_1 + y_2 = 4$ and $y_1 y_2 = 4$

(1) gives,

$$A = \left[\frac{y^2}{2} + y - \frac{y^3}{12}\right]_{y_1}^{y_2}$$

$$= (y_2 - y_1)\left[\frac{y_1 + y_2}{2} + 1 - \frac{y_2^2 + y_1^2 + y_1 y_2}{12}\right]$$



$$A = \left[3 - \frac{(y_2 + y_1)^2 - y_1 y_2}{12}\right]\sqrt{(y_2+y_1)^2 - 4y_1y_2}$$

$$= \left[3 - \frac{16+4}{12}\right]\sqrt{16+20} \;=\; \left(4\sqrt{2}\right)\frac{16}{12} \;=\; \frac{16\sqrt{2}}{3} \;=$$

$$\frac{8\sqrt{8}}{3}$$

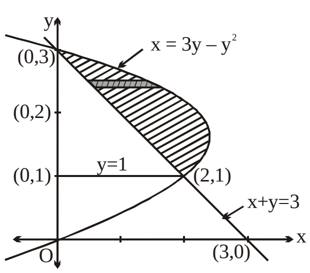
Hence Minimum value of (a + b + c) = 8 + 8 +3 = 19 **Ans.**

Q.7 [1]

Area between $x = 3y - y^2$ and $x + y = 3$

$$A = \int_1^3 \left(3y - y^2\right) - \left(3 - y\right) dy$$

$$= 4 \cdot \frac{y^2}{2} - \frac{y^3}{3} - 3y\Big]_1^3 = (2 \cdot 9 - 9 - 9) - \left(2 - \frac{1}{3} - 3\right)$$



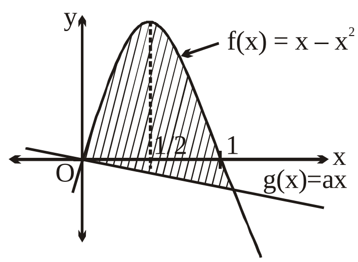
$$A = \frac{4}{3}.$$

Now area bounded by f(x) above the x-axis is = $\int_0^1 \left(x - x^2\right)dx = \frac{1}{6}$, hence a < 0 (think)

Given, $\displaystyle\int_0^{1-a} \left((x - x^2) - (a\,x)\right)dx = \frac{4}{3}$

Solving $\frac{1}{6}\,(1 - a)^3 = \frac{4}{3}$

Note :
$x - x^2 = ax$
$x^2 + (a-1)x = 0$
$x_1 + x_2 = 1 - a$
but $x_1 = 0$
$\Rightarrow x_2 = 1 - a$



$(1 - a)^3 = 8$

$1 - a = 2 \Rightarrow a = -1$

$\Rightarrow$ $|\,[a]\,| = 1$. **Ans.**

Q.8 [0004]

We have $y = |\cos^{-1}(\sin x)| - |\sin^{-1}(\cos x)|$ =

$$\left|\frac{\pi}{2} - \sin^{-1}(\sin x)\right| - \left|\frac{\pi}{2} - \cos^{-1}(\cos x)\right|$$

$$= \left|\frac{\pi}{2} - \left(x - 2\pi\right)\right| - \left|\frac{\pi}{2} - \left(2\pi - x\right)\right| \;=\; \left|\frac{5\pi}{2} - x\right| -$$

$$\left|x - \frac{3\pi}{2}\right| \;=\; \left(\frac{5\pi}{2} - x\right) - \left(x - \frac{3\pi}{2}\right) \;=\; 4\pi - 2x$$

$$\left(\text{For } x \in \left[\frac{3\pi}{2}, 2\pi\right], \sin^{-1}(\sin x) = x - 2\pi \;\text{ and }\; \cos^{-1}(\cos x) = 2\pi - x\right)$$

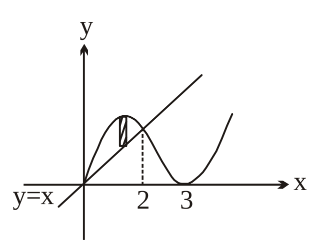
Clearly required area = Area of shaded triangle in figure

$$= \frac{1}{2} \times \left(\frac{\pi}{2}\right) \times \pi \;=\; \frac{\pi^2}{4} = \frac{\pi^2}{k} \text{ (Given)}$$

Hence k = 4 Ans.

Q.9 [8]

$$A = \int_0^2 [x(x-3)^2 - x]dx$$



Q.10 [2]

We have $A(t) = \int_0^t \sin x^2\, dx$; $B(t) = \dfrac{t \sin t^2}{2}$

$$\therefore \lim_{t\to 0} \frac{A(t)}{B(t)} = \lim_{t\to 0} \frac{2\int_0^t \sin x^2\, dx}{t \sin t^2}$$

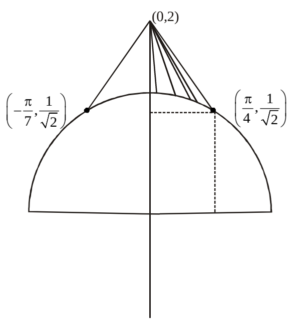
$$= \lim_{t\to 0} \frac{2\int_0^t \sin x^2\, dx}{t^3 \dfrac{\sin t^2}{t^2}} = \lim_{t\to 0} \frac{2\int_0^t \sin x^2\, dx}{t^3}$$

Hence $\lim_{t\to 0} \dfrac{A(t)}{B(t)} = \lim_{t\to 0} \dfrac{2\sin t^2}{3t^2} = \dfrac{2}{3}$

x = 2 **Ans.**

31KVPY
PREVIOUS YEAR'S

Q.1 (A)



required area $= 2\left[\frac{1}{2}\left(\frac{1}{\sqrt{2}}+2\right).\frac{\pi}{4} - \int_0^{\pi/4} \cos x\, dx\right]$

$= \left(\dfrac{4+\sqrt{2}}{8}\right)\pi - \sqrt{2}$

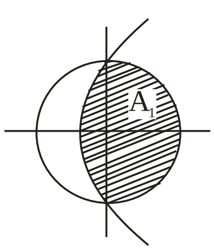
Q.2 (C)

$A = \int_0^1 f(x)\,dx - \int_1^3 f(x)\,dx$

$= \int_0^1 (x^3 - 4x^2 + 3x)dx - \int_1^3 (x^3 - 4x^2 + 3x)dx = \dfrac{37}{12}$

Q.3 (B)

$A_1 = 2\int_{-1/4}^{0} \sqrt{4x+1}\, dx + 2\int_0^1 \sqrt{1-x^2}\, dx$



Solve $A_1 = \dfrac{1}{3} + \dfrac{\pi}{2}$

$A_2 = \pi(1)^2 - A_1 = \dfrac{\pi}{2} - \dfrac{1}{3}$

$|A_1 - A_2| = \dfrac{2}{3}$

Q.4 (B)

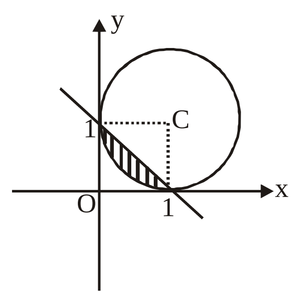
Q.5 (B)

$y + e^x + \dfrac{\sin x}{x}$ and $y = \dfrac{\sin x}{x} + \dfrac{x^2}{2}$

$\text{Area} = \int_1^2 \left[\left(e^x + \frac{\sin x}{x}\right) - \left(\frac{x^2}{2} + \frac{\sin x}{x}\right)\right] dx = \left| e^x - \frac{x^3}{6} \right|_1^2$

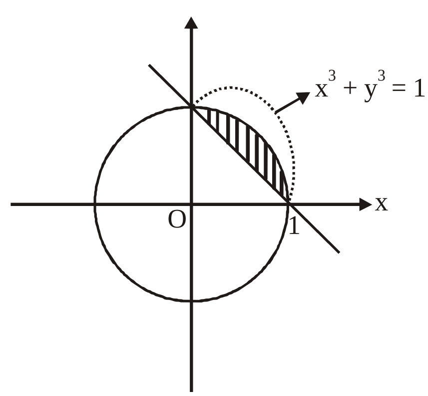
$= e^2 - \dfrac{4}{3} - e + \dfrac{1}{6} = e^2 - e - \dfrac{7}{6}$

Q.6 (C)



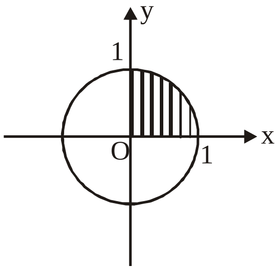
$|A_1| = \dfrac{\pi}{4} - \dfrac{1}{2}$

$|A_2| = \dfrac{\pi}{4} - \dfrac{1}{2}$



$|A_3| > |A_2|$

Q.7 (A)



$x^2 + y^2 \le 1$ and $x, y \in [0, 1]$

$$\int_0^1 f(x)\,dx = \frac{\pi}{4} \Rightarrow \text{area} = \frac{\pi}{4}$$

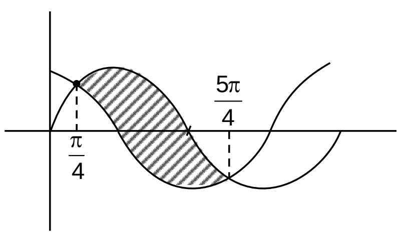
$$\Rightarrow y = \sqrt{1-x^2}$$

$$\Rightarrow \int_{1/2}^{1/\sqrt{2}} \frac{f(x)}{1-x^2}dx = \int_{1/2}^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}} = \left|\sin^{-1}x\right|_{1/2}^{1/\sqrt{2}}$$

$$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

JEE-MAIN
PREVIOUS YEAR'S

Q.1 [64]



$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx = \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$$

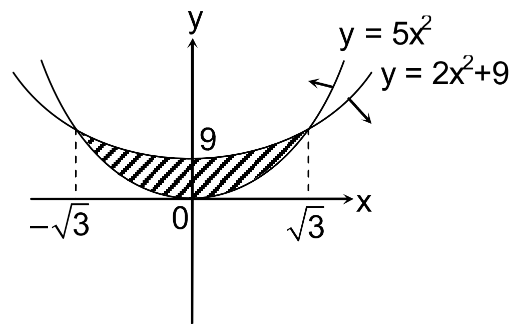
$$= -\left[\left(-\cos\frac{5\pi}{4} + \sin\frac{\pi}{4}\right) - \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right]$$

$$= -\left[\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$\Rightarrow A^4 = (2\sqrt{2})^4 = 64$

Q.2 (2)



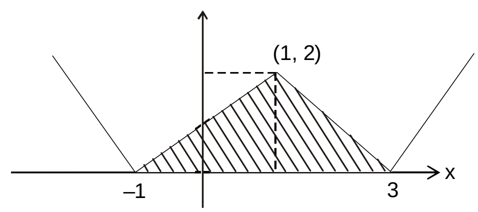
Required area

$$= 2\int_0^{\sqrt{3}} (2x^2 + 9 - 5x^2)dx$$

$$= 2\int_0^{\sqrt{3}} (9 - 3x^2)dx$$

$$= 2\left|9 - 3x^2\right|_0^{\sqrt{3}} = 12\sqrt{3}$$

Q.3 [4]



$$\text{Area} = \frac{1}{2} \times 4 \times 2 = 4$$

Q.4 [9.00]

$$\text{Let } f(x) = \frac{4}{\sin x} + \frac{1}{1 - \sin x}$$

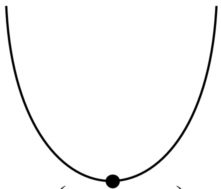
$$y = \frac{4 - 3\sin x}{\sin x(1 - \sin x)}$$

Let $\sin x = t$ when $t \in (0,1)$

$$y = \frac{4 - 3t}{t - t^2}$$

$$\frac{dy}{dt} = \frac{-3(t - t^2) - (1 - 2t)(4 - 3t)}{(t - t^2)^2} = 0$$

$\Rightarrow 3t^2 - 3t - (4 - 11t + 6t^2) = 0$
$\Rightarrow 3t^2 - 8t + 4 = 0$
$\Rightarrow 3t^2 - 6t - 2t + 4 = 0$



$\left(\sin^{-1}\frac{2}{3}, 9\right)$

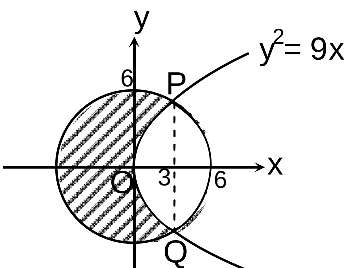
$\Rightarrow t = \frac{2}{3}$

$\Rightarrow \alpha \ge 9$

least α is equal to 9

Q.5 (3)

The curves intersect at points $(3, \pm 3\sqrt{3})$



Required area

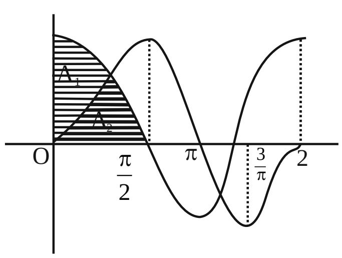
$$= \pi r^2 - 2\left[\int_0^3 \sqrt{9x}\,dx + \int_0^6 \sqrt{36 - x^2}\,dx\right]$$

$$= 36\pi - 12\sqrt{3} - 2\left|\frac{x}{2}\sqrt{36 - x^2} + 18\sin^{-1}\left(\frac{x}{6}\right)\right|_3^6$$

$$= 36 - 12\sqrt{3} - 2\left(9\pi - \left(\frac{9\sqrt{3}}{2} + 3\pi\right)\right) = 24\pi - 3\sqrt{3}$$

Q.6 (1)

$$A_1 + A_2 = \int_0^{\pi/2} \cos x.dx = \sin x|_0^{\pi/2} = 1$$



$$A_1 = \int_0^{\pi/4} (\cos x - \sin x)dx = (\sin x + \cos x)|_0^{\pi/4} = \sqrt{2} - 1$$

$$\therefore A_2 = 1 - (\sqrt{2} - 1) = 2 - \sqrt{2}$$

$$\therefore \frac{A_1}{A_2} = \frac{\sqrt{2} - 1}{\sqrt{2}(\sqrt{2} - 1)} = 1 : \sqrt{2}$$

Q.7 [2]

$f(-1) = 3 > 0$

$f(-2) = -64 + 80 - 80 + 40 - 20 + 10$

$= -34 < 0$

$\therefore$ At least one root in $(-2, -1)$

$f'(x) = 10(x^4 + 2x^3 + 3x^2 + 2x + 1)$

$$= 10\left(x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) + 3\right)$$

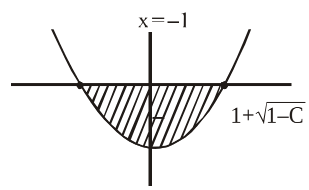
$$= 10\left(\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) + 1\right)$$

$$= 10\left(\left(x + \frac{1}{x}\right) + 1\right)^2 > 0; \ \forall x \in R$$

$\therefore$ Exactly one real root in $(-2, -1)$

Q.8 [2]

$$\frac{dy}{dx} = 2(x+1)$$



$\Rightarrow \int dy = \int 2(x+1)dx$

$\Rightarrow y(x) = x^2 + 2x + C$

$$\text{Area} = \frac{4\sqrt{8}}{3}$$

$-1 + \sqrt{1 - C}$

$$\Rightarrow 2\int_{-1}^{-1+\sqrt{1-C}} (-(x+1)^2 - C + 1)dx = \frac{4\sqrt{8}}{3}$$

$$\Rightarrow 2\left[-\frac{(x+1)^3}{3} - Cx + x\right]_{-1}^{-1+\sqrt{1-C}} = \frac{4\sqrt{8}}{3}$$

$\Rightarrow -(\sqrt{1-C})^3 + 3c - 3C\sqrt{1-C}$

$-3 + 3\sqrt{1-C} - 3C + 3 = 2\sqrt{8}$

$\Rightarrow C = -1$

$\Rightarrow f(x) = x^2 + 2x - 1, \ f(1) = 2$

Q.9 (1)

$$f(x) = 3\ell n(x-1) - 3\ell n(x+1) - \frac{2}{x-1}$$

$$f'(x) = \frac{3}{x-1} - \frac{3}{x+1} + \frac{2}{(x-1)^2}$$

$$f'(x) = \frac{4(2x-1)}{(x-1)^2(x+1)}$$

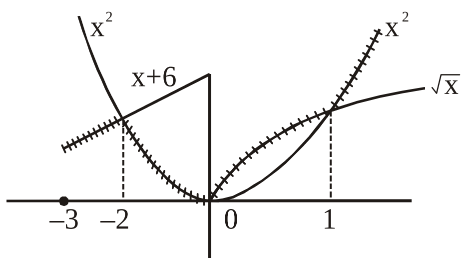
$f'(x) \geq 0$

$$\Rightarrow x \in (-\infty, -1) \cup \left[\frac{1}{2}, 1\right) \cup (1, \infty)$$

Q.10 [41]

f : [–3, 1] → R

$$f(x) = \begin{cases} \min\{(x+6), x^2\}, & -3 \leq x \leq 0 \\ \max\{\sqrt{x}, x^2\}, & 0 \leq x \leq 1 \end{cases}$$



area bounded by y = f(x) and x-axis

$$= \int_{-3}^{-2} (x+6)dx + \int_{-2}^{0} x^2 dx + \int_{0}^{1} \sqrt{x}\, dx$$

$$A = \frac{41}{6}$$

6A = 41

Q.11 (3)

$4y^2 = x^2(4-x)(x-2)$

$$|y| = \frac{|x|}{2}\sqrt{(4-x)(x-2)}$$

$$\Rightarrow y_1 = \frac{x}{2}\sqrt{(4-x)(x-2)}$$

and $y_2 = \frac{-x}{2}\sqrt{(4-x)(x-2)}$

D : x ∈ [2, 4]

Required Area

$$= \int_2^4 (y_1 - y_2)\, dx = \int_2^4 x\sqrt{(4-x)(x-2)}\, dx \quad ...(1)$$

Applying $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$\text{Area} = \int_2^4 (6-x)\sqrt{(4-x)(x-2)}\, dx \quad ...(2)$$

(1) + (2)

$$2A = 6\int_2^4 \sqrt{(4-x)(x-2)}\, dx$$

$$A = 3\int_2^4 \sqrt{1-(x-3)^2}\, dx$$

$$A = 3.\frac{\pi}{2}.1^2 = \frac{3\pi}{2}$$

Q.12 [4]

$$xdy - ydx = \sqrt{x^2 - y^2}\, dx$$

$$\Rightarrow \frac{xdy - ydx}{x^2} = \frac{1}{x}\sqrt{1 - \frac{y^2}{x^2}}\, dx$$

$$\Rightarrow \int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1-\left(\frac{y}{x}\right)^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \ell n|x| + c$$

at x = 1, y = 0 ⇒ c = 0

y = xsin(lnx)

$$A = \int_1^{e^\pi} x\sin(\ln x)dx$$

$$x = e^t,\ dx = e^t dt \Rightarrow \int_0^\pi e^{2t}\sin(t)\, dt = A$$

$$\alpha e^{2\pi} + \beta = \left(\frac{e^{2t}}{5}(2\sin t - \cos t)\right)_0^\pi = \frac{1+e^{2\pi}}{5}$$

$\alpha = \frac{1}{5}$ $\beta = \frac{1}{5}$ so $10(\alpha + \beta) = 4$

Q.13 [2]

Q.14 (4)

Q.15 (2)

Q.16 (3)

Q.17 (1)

Q.18 [26]

Q.19 [114]

Q.20 [27]

Q.21 (2)

JEE-ADVANCED PREVIOUS YEAR'S

Q.1 **(ABD)**

$$I = \int_0^1 e^{-x^2} dx$$

$-x^2 \le 0$

$e^{-x^2} \le 1$

$$\int_0^1 e^{-x^2} dx \le 1$$

$x^2 \le x \Rightarrow -x^2 \ge -x \Rightarrow e^{-x^2} \ge e^{-x}$

$$\Rightarrow I \ge \int_0^1 e^{-x} dx$$

$$\ge -\left(e^{-x}\right)_0^1$$

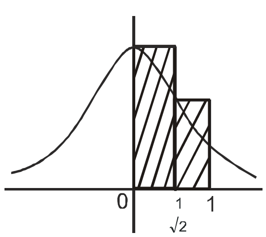
$$\ge -\left(\frac{1}{e} - 1\right)$$

$I \ge 1 - \frac{1}{e} \Rightarrow$ (B) is correct

Since If $I \ge 1 - \frac{1}{e} \Rightarrow I > \frac{1}{e}$

$\Rightarrow$ (A) is correct

$I < \frac{1}{\sqrt{2}} \times 1 + \frac{1}{\sqrt{e}} \times (1 - \frac{1}{\sqrt{2}})$



So Ans. D

[Hence Answers are A, B and D]

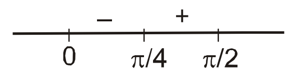
Q.2 (B)

Given` $y = \sin x + \cos x \;\; x \in [0, \pi/2]$

$$\frac{dy}{dx} = \cos x - \sin x$$

$y = |\cos x - \sin x| =$

$$\begin{cases} \cos x - \sin x & x \in [0, \pi/4] \\ \sin x - \cos x & x \in [\pi/4, \pi/2] \end{cases}$$



required area =

$$\int_0^{\pi/4} |(\sin x + \cos x) - (\cos x - \sin x)| dx + \int_{\pi/4}^{\pi/2} |2\cos x| dx$$

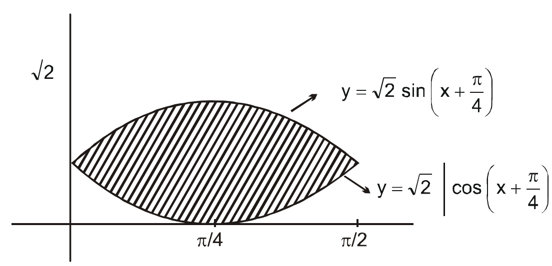
$$= \int_0^{\pi/4} |2\sin x| dx + \int_{\pi/4}^{\pi/2} |2\cos x| dx =$$

$$2(-\cos x)_0^{\pi/4} + 2(\sin x)_{\pi/4}^{\pi/2} =$$

$$2\left[-\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}}\right]$$

$$= 2\left(2 - \frac{2}{\sqrt{2}}\right)$$

$$= 2\left(2 - \sqrt{2}\right)$$



$= 4 - 2\sqrt{2}$

$= 2\sqrt{2}\left(\sqrt{2} - 1\right)$

Q.3 (AC)

It may be discontinuous at x = a or x = b

$$\lim_{x \to a^-} g(x) = 0$$

$$\lim_{x \to a^+} g(x) = \lim_{x \to a^+} \int_a^x f(t)dt = \int_a^a f(t)dt = 0$$

$$g(a) = \int_a^a f(t)dt = 0$$

Similarly at x = b we will get continuous

So g(x) is continuous $\forall\, x \in R$

$$g'(x) = \begin{cases} 0 & x < a \\ f(x) & a \le x \le b \\ 0 & x > b \end{cases}$$

$g'(a^-) = 0$ $\qquad$ $g'(b^-) = f(b)$

$g'(a^+) = f(a)$ $\qquad$ $g'(b^+) = 0$

Since f(x) co-domain is $[1, \infty)$ f(a) & f(b) can never be zero.

Hence it is non derivable at x = a & x = b.

Q.4 (AC)

$y = x^3$

$$\int_0^1 \left(x - x^3\right) dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\int_0^\alpha \left(x - x^3\right) dx = \frac{1}{8}$$

$4\alpha^2 - 2\alpha^4 = 1$

$2\alpha^4 - 4\alpha^2 + 1 = 0$

$2t^2 - 4t + 1 = 0$ (taking $t = \alpha^2$)

$$t = \frac{4 \pm \sqrt{16-8}}{4}$$

$$t = \frac{4 \pm 2\sqrt{2}}{4}$$

$$t = \alpha^2 = 1 \pm \frac{1}{\sqrt{2}}$$

$$\therefore\ \alpha^2 = 1 - \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \frac{1}{2} < \alpha < 1$$

Q.5 (B,C)

$$f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)\,dt$$

$$\Rightarrow e^{-x} f(x) = e^{-x}(1 - 2x) + \int_0^x e^{-t} f(t)\,dt$$

Differentiate w.r.t. x.

$-e^{-x} f(x) + e^{-x} f'(x) = -e^{-x}(1 - 2x) + e^{-x}(-2) + e^{-x} f(x)$

$\Rightarrow -f(x) + f'(x) = -(1 - 2x) - 2 + f(x).$

$\Rightarrow f'(x) - 2f(x) = 2x - 3$

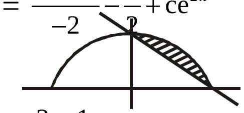
Integrating factor $= e^{-2x}$.

$$f(x).e^{-2x} = \int e^{-2x}(2x - 3)\,dx$$

$$= (2x - 3)\int e^{-2x}\,dx - \int \left((2)\int e^{-2x}dx\right) dx$$

$$= \frac{(2x-3)e^{-2x}}{-2} - \frac{e^{-2x}}{2} + c$$

$$f(x) = \frac{2x-3}{-2} - \frac{1}{2} + ce^{2x}$$

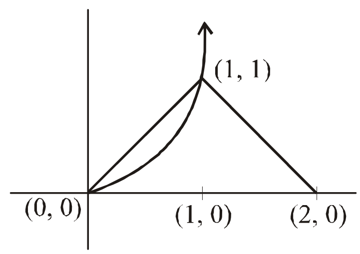


$$f(0) = \frac{3}{2} - \frac{1}{2} + c = 1 \quad \Rightarrow \quad c = 0$$

$\therefore$ $\qquad$ $f(x) = 1 - x$

$$\text{Area} = \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}$$

Q.6 [4]



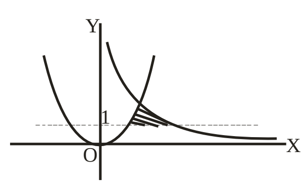
$$\text{Area} = \int_0^1 (x - x^n)\,dx = \frac{3}{10}$$

$$\left[\frac{x^2}{2} - \frac{x^{x+1}}{n+1}\right]_0^1 = \frac{3}{10}$$

$$\frac{1}{2} - \frac{1}{n+1} = \frac{3}{10} \qquad \therefore\ n + 1 = 5$$

$\Rightarrow$ n = 4

Q.7 (B)



For intersection, $\frac{8}{y} = \sqrt{y} \Rightarrow y = 4$

Hence, required area $= \int_1^4 \left(\frac{8}{y} - \sqrt{y}\right) dy$

$= \left[8\ell n\, y - \frac{2}{3}y^{3/2}\right]_1^4 = 16\ell n^2 - \frac{14}{3}$

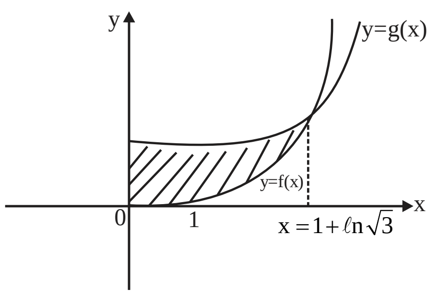
Remark : The question should contain the phrase "area of the bounded region in the first quadrant". Because, in the 2nd quadrant, the region above the line y = 1 and below $y = x^2$, satisfies the region, which is unbounded.

Q.8 (A)

Here,

$f(x) = \begin{cases} 0 & x \le 1 \\ e^{x-1} - e^{1-x} & x \ge 1 \end{cases}$

& g(x) $= \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)$



solve f(x) & g(x) $\Rightarrow x = 1 + \ell n\sqrt{3}$

So bounded area =

$$\int_0^1 \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)dx + \int_1^{1+\ell n\sqrt{3}} \frac{1}{2}\left(e^{x-1} + e^{1-x}\right) - \left(e^{x-1} + e^{1-x}\right)dx$$

$$= \frac{1}{2}\left[e^{x-1} - e^{1-x}\right]_0^1 + \left[-\frac{1}{2}e^{x-1} - \frac{3}{2}e^{1-x}\right]_1^{1+\ell n\sqrt{3}}$$

$$= \frac{1}{2}\left[e - \frac{1}{e}\right] + \left[\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + 2\right] = 2 - \sqrt{3} + \frac{1}{2}\left(e - \frac{1}{e}\right)$$

Differential Equation

EXERCISES

ELEMENTRY

Q.1 (1)

Given differential equation can be written as

$$y^2+x^2\left(\frac{dy}{dx}\right)^2-2xy.\frac{dy}{dx}=a^2\left(\frac{dy}{dx}\right)^2+b^2$$

Hence it is of 1st order and 2nd degree differential equation.

Q.2 (2)

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/4}=\left(\frac{d^2y}{dx^2}\right)^{1/3}$$

$$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^{9}=\left(\frac{d^2y}{dx^2}\right)^{4}$$

Clearly, degree is 4.

Q.3 (3)

Let $y=4\sin 3x \Rightarrow \frac{dy}{dx}=12\cos 3x$

$$\Rightarrow \frac{d^2y}{dx^2}=-36\sin 3x=-9\times 4\sin 3x=-9y$$

$$\Rightarrow \frac{d^2y}{dx^2}+9y=0$$

Q.4 (1)

$$y=A\sin x+B\cos x \Rightarrow \frac{dy}{dx}=A\cos x-B\sin x$$

$$\Rightarrow \frac{d^2y}{dx^2}=-A\sin x-B\cos x=-(A\sin x+B\cos x)=-y$$

$\Rightarrow \frac{d^2y}{dx^2}+y=0$ is the required differential equation.

Q.5 (2)

Given equation $y=a\cos(x+b)$

Differentiate it *w.r.t. x* we get $\frac{dy}{dx}=-a\sin(x+b)$

Again $\frac{d^2y}{dx^2}=-a\cos(x+b)=-y$ or $\frac{d^2y}{dx^2}+y=0$.

Q.6 (1)

$x^2y=a$ (On differentiating)

$$x^2\frac{dy}{dx}+y\frac{d}{dx}(x^2)=0 \Rightarrow x^2\frac{dy}{dx}+2xy=0$$

$$\Rightarrow \frac{dy}{dx}+\frac{2y}{x}=0$$

Q.7 (1)

It can be written in the form of

$$\frac{\sec^2 y}{\tan y}dy=-3\frac{e^x}{1-e^x}dx$$

$$\int\frac{\sec^2 y}{\tan y}dy=-3\int\frac{e^x}{1-e^x}dx$$

$$\Rightarrow \log(\tan y)=3\log(1-e^x)+\log c$$

$$\Rightarrow \tan y=c(1-e^x)^3$$

Q.8 (2)

$$\frac{dy}{dx}=-\frac{\cos x-\sin x}{\sin x+\cos x}$$

$$\Rightarrow dy=-\left(\frac{\cos x-\sin x}{\sin x+\cos x}\right)dx$$

On integrating both sides, we get

$$\Rightarrow y=-\log(\sin x+\cos x)+\log c$$

$$\Rightarrow y=\log\left(\frac{c}{\sin x+\cos x}\right)$$

$$\Rightarrow e^y(\sin x+\cos x)=c.$$

Q.9 (4)

$$\frac{dy}{dx}=(1+x)(1+y^2) \Rightarrow \frac{dy}{1+y^2}=(1+x)dx$$

On integrating both sides, we get

$$\tan^{-1}y=\frac{x^2}{2}+x+c \Rightarrow y=\tan\left(\frac{x^2}{2}+x+c\right)$$

Q.10 (2)

$$x\sec y\frac{dy}{dx}=1 \Rightarrow \sec y dy=\frac{dx}{x}$$

On integrating both sides, we get

$$\log(\sec y+\tan y)=\log x+\log c$$

$$\Rightarrow \sec y+\tan y=cx.$$

Q.11 (1)

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}} \Rightarrow dy=-\frac{1}{\sqrt{1-x^2}}dx$$

On integrating, we get $y=\cos^{-1}x+c$

$$\Rightarrow y=\frac{\pi}{2}-\sin^{-1}x+c \Rightarrow y+\sin^{-1}x=c$$

Q.12 (3)

Let $x-y=v$ and $\frac{dy}{dx}=1-\frac{dv}{dx}$, thus the equation reduces to $\frac{dv}{dx}=\frac{v+2}{2v+5} \Rightarrow \int \frac{2v+5}{v+2}dv=\int dx$

$$\Rightarrow \int\left[2+\frac{1}{(v+2)}\right]dv=\int dx$$

$$\Rightarrow 2v+\log(v+2)=x+c$$

$$2(x-y)+\log(x-y+2)=x+c$$

Q.13 (1)

Given equation is,

$$(x\sqrt{1+y^2})dx+(y\sqrt{1+x^2})dy=0$$

$$\Rightarrow x\sqrt{1+y^2}dx=-y\sqrt{1+x^2}dy$$

$$\Rightarrow \int \frac{x}{\sqrt{1+x^2}}dx+\int \frac{y}{\sqrt{1+y^2}}dy=c$$

$$\Rightarrow \sqrt{1+x^2}+\sqrt{1+y^2}=c$$

Q.14 (3)

Put $x+y=v$ and $1+\frac{dy}{dx}=\frac{dv}{dx}$

$$\Rightarrow \frac{dv}{dx}=v^2+1 \Rightarrow \frac{dv}{v^2+1}=dx$$

On integrating, we get

$\tan^{-1}v=x+c$ or

$v=\tan(x+c) \Rightarrow x+y=\tan(x+c)$

Q.15 (2)

Given $\sin\frac{dy}{dx}=a$; $dy=\sin^{-1}a\,dx$

Integrating both sides, $\int dy=\int \sin^{-1}a\,dx$

$y=x\sin^{-1}a+c$ and $y(0)=0+c=1$, $\therefore c=1$

$$\therefore y=x\sin^{-1}a+1 \Rightarrow a=\sin\frac{y-1}{x}.$$

Q.16 (4)

Put $x+y=v \Rightarrow 1+\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-1$

$$\therefore v^2\left(\frac{dv}{dx}-1\right)=a^2$$

$$\Rightarrow \frac{dv}{dx}=\frac{a^2}{v^2}+1=\frac{a^2+v^2}{v^2} \Rightarrow \frac{v^2}{a^2+v^2}dv=dx$$

$$\Rightarrow \left(1-\frac{a^2}{a^2+v^2}\right)dv=dx \Rightarrow v-a\tan^{-1}\frac{v}{a}=x+c$$

$$\Rightarrow y=a\tan^{-1}\left(\frac{x+y}{a}\right)+c.$$

Q.17 (1)

$$\frac{dy}{dx}=\frac{x\log x^2+x}{\sin y+y\cos y}.$$

Separating the variables and integrating

$$\int(\sin y+y\cos y)dy=\int(x\log x^2+x)dx$$

$$\Rightarrow -\cos y+y\sin y+\cos y$$

$$=\frac{x^2}{2}\log x^2-\int\frac{x^2}{2}.\frac{1}{x^2}.2xdx+\int x\,dx+c$$

$$\Rightarrow y\sin y=\frac{x^2}{2}2\log x-\int x\,dx+\int xdx+c$$

$$\Rightarrow y\sin y=x^2\log x+c$$

Q.18 (1)

$$\frac{dy}{dx}\tan y=\sin(x+y)+\sin(x-y)$$

$$\frac{dy}{dx}(\tan y)=2\sin x\cos y \Rightarrow \frac{\sin y}{\cos^2 y}dy=2\sin xdx$$

$$\Rightarrow \int\frac{\sin y}{\cos^2 y}dy=2\int\sin xdx \Rightarrow \frac{1}{\cos y}=-2\cos x+c$$

$$\therefore \sec y+2\cos x=c$$

Q.19 (4)

It is homogeneous equation

$$\frac{dy}{dx}=\frac{x^2+3y^2}{2xy}$$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So, we get $x\frac{dv}{dx} = \frac{1+v^2}{2v}$

$\Rightarrow \frac{2vdv}{1+v^2} = \frac{dx}{x}$

On integrating, we get $x^2 + y^2 = px^3$

Q.20 (1)

$\frac{dy}{dx} = \frac{x}{2y-x}$. Put $y = vx \Rightarrow v + x\frac{dv}{dx} = \frac{dy}{dx}$

$v + x\frac{dv}{dx} = \frac{x}{2v-x} = \frac{1}{2v-1}$

$x\frac{dv}{dx} = \frac{1}{2v-1} - v = \frac{1-2v^2+v}{2v-1} = -\frac{(v-1)(2v+1)}{2v-1}$

$\frac{(2v-1)}{(2v+1)(v-1)} = \frac{-dx}{x}$; $\frac{1}{3(v-1)} + \frac{4}{3(2v+1)} = \frac{-dx}{x}$

$\Rightarrow \frac{1}{3}\log(v-1) + \frac{4}{3}.\frac{1}{2}\log(2v+1) = \log\frac{1}{x} + \log c$

$\Rightarrow \log(v-1)^{1/3} + \log(2v+1)^{2/3} = \log\frac{c}{x}$

$\Rightarrow (v-1)^{1/3}(2v+1)^{2/3} = \frac{c}{x}$

$\Rightarrow \left(\frac{y-x}{x}\right)\left(\frac{2y+x}{x}\right)^2 = \frac{c^3}{x^3} \Rightarrow (x-y)(x+2y)^2 = c$

Q.21 (1)

Given $\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x.\frac{dv}{dx}$

$\therefore v + x.\frac{dv}{dx} = v(\log v + 1)$

$v + x\frac{dv}{dx} = v\log v + v \Rightarrow x\frac{dv}{dx} = v\log v$

$\Rightarrow \frac{dv}{v\log v} = \frac{dx}{x}$

Integrating both sides, $\int \frac{dv}{v\log v} = \int \frac{dx}{x}$

$\log\log v = \log x + \log c \Rightarrow \log v = xc$

$\Rightarrow \log(y/x) = xc$.

Q.22 (1)

$ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

$\Rightarrow \frac{ydx - xdy}{y^2} + 3x^2e^{x^3}dx = 0$

$\Rightarrow d\left(\frac{x}{y}\right) + de^{x^3} = 0$

On integrating, we get $\frac{x}{y} + e^{x^3} = c$

Q.23 (2)

$xdx - y^3dx + 3xy^2dy = 0$

Put $y^3 = t \Rightarrow dt = 3y^2dy$

$x\,dx - tdx + xdt = 0 \Rightarrow xdx + xdt - tdx = 0$

$\Rightarrow \frac{dx}{x} + d\left(\frac{t}{x}\right) = 0$

On integration, we get $\log x + \frac{t}{x} = k$

$\Rightarrow \log x + \frac{y^3}{x} = k$

Q.24 (1)

$ye^{-x/y}dx - (xe^{-x/y} + y^3)dy = 0$

$\Rightarrow e^{-x/y}(ydx - xdy) = y^3dy$

$\Rightarrow e^{-x/y}\frac{(ydx - xdy)}{y^2} = ydy$

$\Rightarrow e^{-x/y}d\left(\frac{x}{y}\right) = ydy$. Integrating both sides, we get

$\Rightarrow k - e^{-x/y} = \frac{y^2}{2} \Rightarrow \frac{y^2}{2} + e^{-x/y} = k$

Q.25 (2)

$(1+y^2)dx - (\tan^{-1}y - x)dy = 0$

$\Rightarrow \frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}$

$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$

$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$

This is equation of the form

$$\frac{dx}{dy}+Px=Q$$

So, I.F. $=e^{\int P\,dy}=e^{\int\frac{1}{1+y^2}.dy}=e^{\tan^{-1}y}$.

Q.26 (1)

The given equation $\frac{dy}{dx}+\frac{y}{x}=x^2$ is of the form

$\frac{dy}{dx}+Py=Q$. So, I.F.= $e^{\int\frac{1}{x}dx}=e^{\log x}=x$

Hence required solution $xy=\int x.x^2dx+c$

$\Rightarrow xy=\frac{x^4}{4}+c \Rightarrow 4xy=x^4+c$.

Q.27 (2)

$x\frac{dy}{dx}+3y=x \Rightarrow \frac{dy}{dx}+\frac{3y}{x}=1$

It is in the form of $\frac{dy}{dx}+Py=Q$

So, I.F. $=e^{\int Pdx}=e^{3\int\frac{1}{x}dx}=e^{3\log x}=x^3$

Hence required solution is

$y+x^2+2x+2=ce^x \Rightarrow yx^3=\frac{x^4}{4}+c$.

Q.28 (1)

$y+x^2=\frac{dy}{dx} \Rightarrow \frac{dy}{dx}-y=x^2$

This is the linear differential equation in *y*, where

$P=-1, Q=x^2$

I.F. $=e^{\int P.dx}=e^{\int -dx}=e^{-x}$

Hence solution, $y.(\text{I.F}).=\int Q.(\text{I.F})\,dx+c$

$\Rightarrow ye^{-x}=-x^2e^{-x}-2xe^{-x}-2e^{-x}+c$

$\Rightarrow y+x^2+2x+2=ce^x$.

Q.29 (1) $\frac{dy}{dx}=\frac{-2xy}{(x^2+1)} \Rightarrow \frac{dy}{y}=-\frac{2x}{x^2+1}dx$

On integrating, we get

$\log y=-\log(1+x^2)+\log c \Rightarrow y(1+x^2)=c$

Since curve passes through (1, 2), we have

$c=2(1+1^2) \Rightarrow c=4$

Hence solution is $y(x^2+1)=4$.

Q.30 (1)

Slope $=\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{y-1}{x^2+x} \Rightarrow \frac{dy}{y-1}=\frac{dx}{x^2+x}$

$\Rightarrow \int \frac{1}{y-1}dy=\int\left(\frac{1}{x}-\frac{1}{x+1}\right)dx+c$

$\Rightarrow \frac{(y-1)(x+1)}{x}=k$

Putting $x=1$, $y=0$ we get $k=-2$

Hence the equation is $(y-1)(x+1)+2x=0$.

JEE-MAIN OBJECTIVE QUESTIONS

Q.1 (1)

$$\left(\frac{d^2y}{dx^2}\right)^2 r^2=\left[1+\frac{dy}{dx}\right]^3$$

order : 2
degree : 2

Q.2 (4)

$y=k_1\sin(x+C_3)-k_2e^x$

$k_1 : C_1+C_2$; $k_2=c_4e^{C_5}$

order : 3

Q.3 (1)

tangent to $x^2=4y$

$x=my+\frac{1}{m}$

$m=\frac{dy}{dx} \Rightarrow x=y\left(\frac{dy}{dx}\right)+\frac{1}{(dy/dx)}$

$\Rightarrow x\left(\frac{dy}{dx}\right)=y\left(\frac{dy}{dx}\right)^2+1$

$\therefore$ order = 1
degree = 2

Q.4 (4)

$$y^2\left(\frac{d^2y}{dx^2}\right)^2+x^2y^2-\sin x=-3x\left(\frac{dy}{dx}\right)^{1/3}$$

$$\left(y^2\left(\frac{d^2y}{dx^2}\right)^2+x^2y^2-\sin x\right)^3=-9x^3\left(\frac{dy}{dx}\right)$$

here order = 2 = p
Degree = 6 = q
$\therefore$ p < q

Q.5 (1)

$y = Ax + A^3 \Rightarrow \frac{dy}{dx} = A$

$\therefore\ y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3$

Degree = 3

Q.6 (4)

$\left(1+3\frac{dy}{dx}\right)^{2/3} = 4\frac{d^3y}{dx^3}$

$\left(1+\frac{3dy}{dx}\right)^2 = \left(4\frac{d^3y}{dx^3}\right)^3$

order = 3
Degree = 3

Q.7 (2)
$y^2 = 4ax + k$

$2y\frac{dy}{dx} = 4a$

$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$

degree = 1
order = 2

Q.8 (3)

$\left(\frac{dy}{dx}\right)^{1/3} = 4\frac{d^2y}{dx^2} + 7x$

$\left(\frac{dy}{dx}\right) = \left(4\frac{d^2y}{dx^2}+7x\right)^3$

order =2 = a
degree = 3 = b
a + b = 5

Q.9 (2)
$ax^2 + 2hxy + by^2 = 1$
order : 3

Q.10 (2)
$Ax^2 + By^2 = 1$
$2Ax + 2Byy' = 0$

$Ax + Byy' = 0 \Rightarrow \frac{A}{B} = \frac{-yy'}{x}$

$A + B(y')^2 + Byy'' = 0$

$\frac{A}{B} + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$

$\frac{-y}{x}\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$

Order = 2
Degree = 1

Q.11 (3)
$y = e^{mx}$
$D^3y - 3D^2y - 4Dy + 12y = 0$
$m^3e^{mx} - 3m^2e^{mx} - 4me^{mx} + 12e^{mx} = 0$
$m^3 - 3m^2 - 4m + 12 = 0$
$m^2(m-3) - 4(m-3) = 0$
$m = 3, 2, -2$
Two Natural number of m possible

Q.12 (4)
$y = mx + c$
$y' = m$
$D^2y - 3Dy - 4y = -4x$
$0 - 3m - 4(mx + c) = -4x$
$-3m - 4mx - 4C = -4x$
$-4m = -4 \Rightarrow m = 1$

$-3m - 4C = 0 \Rightarrow 4C = -3m \Rightarrow C = -\frac{3}{4}$

Q.13 (3)

$\frac{dy}{dx} = e^{-2y} \Rightarrow \frac{e^{2y}}{2} = x + c$

$y = 0, x = 5 \Rightarrow c = -\frac{9}{2}$

$y(x_0) = 3$

$\Rightarrow \frac{e^6}{2} = x_0 - \frac{9}{2} \Rightarrow x_0 = \frac{e^6+9}{2}$

Q.14 (3)
Let equation of St. Line
$Y - y = m(X - x)$

Distance from origin $\Rightarrow \left|\frac{mx-y}{\sqrt{1+m^2}}\right| = 1$

$\therefore\ (mx - y)^2 = 1 + m^2$

$\left(y-\frac{dy}{dx}x\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

Q.15 (1)

$y' = \frac{x^2+y^2}{x^2-y^2}$

$y'_{(1,2)} = \frac{1+4}{1-4} = \frac{-5}{3}$

Q.16 (2)

$y = a + bx + ce^{-x}$

$y' = b - ce^{-x}$

$y'' = ce^{-x}$

$y''' = -ce^{-x}$

$y''' = -y'' \Rightarrow y''' + y'' = 0$

Q.17 (1)

$(x - h)^2 + (y - k)^2 = a^2$

$(x - h) + (y - k)\,y' = 0 \Rightarrow y' = \frac{-(x-h)}{(y-k)}$

$1 + (y - k)\,y'' + (y)^2 = 0 \Rightarrow y'' = \frac{-a^2}{(y+k)^3}$

(1) option satisfy the given conditions

Q.18 (3)

$y = e^{(k + 1)x}$

$y' = (k + 1)e^{(k + 1)x}$

$y'' = (k + 1)^2\, e^{(k + 1)x}$

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

$(k + 1)^2 - 4\,(k + 1) + 4 = 0$

$k^2 + 2k + 1 - 4k = 0$

$(k - 1)^2 = 0$

$k = 1$

Q.19 (1)

$x^2 + y^2 - 2ay = 0 \;\Rightarrow\; a = \frac{(x^2+y^2)}{2y}$

$2x + 2yy' - 2ay' = 0$

$x + yy' - \left(\frac{x^2+y^2}{2y}\right)y' = 0$

$x + y'\left(\frac{y^2-x^2}{2y}\right) = 0$

$2xy + y'\,(y^2 - x^2) = 0$

$y'\,(x^2 - y^2) = 2xy$

Q.20 (3)

$\frac{dy}{dx} - ky = 0, \qquad \frac{dy}{y} = kdx$

$\ell ny = kx + c$

at $x = 0,\ y = 1\ \therefore\ C = 0$

Now $\ell ny = kx$

$y = e^{kx}$

$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{kx} = 0$

$\therefore\ k < 0$

Q.21 (2)

$L_{SN} = y\frac{dy}{dx}$

$y\frac{dy}{dx} = c$

$\Rightarrow \frac{y^2}{2} = Cx + C_2$

Q.22 (2)

$\frac{dy}{dx} = 1 + x + y + xy = (1 + x)(1 + y)$

$\Rightarrow \int\frac{dy}{1+y} = \int(1+x)dx$

$\Rightarrow \ell n(1 + y) = x + \frac{x^2}{2} + c$

$y(-1) = 0 \Rightarrow c = \frac{1}{2}$

$\ell n(1 + y) = x + \frac{x^2}{2} + \frac{1}{2} = \frac{(1+x)^2}{2}$

$\Rightarrow y = e^{\frac{(1+x)^2}{2}} - 1$

Q.23 (2)

$x\,dy = y\,dx$

$\frac{dy}{y} \Rightarrow \frac{dx}{x} \Rightarrow \ell n\, y - \ell n\, x = c$

$y = kx$

$\therefore$ straight line passing through origin

Q.24 (2)

$ydy = (1 - x)\,dx$

$\frac{y^2}{2} = x - \frac{x^2}{2} + C$

$x^2 + y^2 - 2x - C = 0$

Q.25 (1)

$y\ell ny + xy' = 0$

$y\ell ny + x\frac{dy}{dx} = 0 \Rightarrow \frac{dx}{x} + \frac{dy}{y\ell ny} = 0$

$\ell nx + \ell n\,(\ell ny) = \ell nC$

$x(\ell ny) = C$
$y(1) = e$
$\ell ne = C \Rightarrow C = 1$ $\quad\quad x(\ell ny) = 1$

Q.26 (2)

$\frac{dy}{dx} = 100 - y$

$-\ell n\,(100 - y) = x + C$; $y(0) = 50$

$-\ell n\,(100 - y) = x - \ell n\,50 \Rightarrow C = -\ell n\,50$

$\ell n\left(\frac{100-y}{50}\right) = -x$

$100 - y = 50\,e^{-x}$

$y = 100 - 50\,e^{-x}$

Q.27 (2)

$\phi(x) = \phi'(x)$ $\quad \phi(1) = 2$

$\frac{d\phi}{dx} = \phi$

$\ell n\,\phi(x) = x + c$

$\ell n\,2 = 1 + c \Rightarrow c = \ell n2 - 1$

$\ell n\phi(3) = 3 + c = 2 + \ell n2$

$\Rightarrow \phi(3) = 2e^2$

Q.28 (2)

$\frac{dy}{dx} = \frac{y}{x}\left(\ell n\left(\frac{y}{x}\right) + 1\right)$

Put $y = tx \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$t + x\frac{dt}{dx} = t(\ell nt + 1)$

$x\frac{dt}{dx} = t\,\ell n\,t$

$\frac{dt}{t\ell nt} - \frac{dx}{x} = 0$

$\ell n\,(\ell n\,t) - \ell n\,x = C$

$\frac{\ell nt}{x} = k$

$\ell nt = kx$

$\ell n\left(\frac{y}{x}\right) = kx$

Q.29 (4)

$ydx + xdy + x(xy)\,dy = 0$

Let $xy = t \Rightarrow x = \frac{t}{y}$

$xdy + ydx = dt$

$dt + \left(\frac{t}{y}\right)t\,dy = 0$

$\frac{dt}{t^2} + \frac{dy}{y} = 0 \Rightarrow -\frac{1}{t} + \ell ny = C$

$\frac{-1}{xy} + \ell ny = C$

Q.30 (1)

$\frac{dv}{dt} + \frac{K}{m}v = -g$

Integrating factor (I.F.) $= e^{\int \frac{k}{m}dt} = e^{\frac{K}{m}t}$

$\therefore V\,e^{\frac{K}{m}t} = -\int g\,e^{K\cdot t/m}$

$V\,e^{\frac{K}{m}t} = \frac{-gm}{K}e^{\frac{K}{m}t} + c$

$V = C\cdot e^{\frac{-K}{m}t} - \frac{mg}{K}$

Q.31 (3)

$(2x - 10y^3)\frac{dy}{dx} + y = 0$

$\frac{dy}{dx} = \frac{y}{10y^3 - 2x} \Rightarrow \frac{dx}{dy} = \frac{10y^3 - 2x}{y}$

$\frac{dx}{dy} + \frac{2x}{y} = 10y^2$

I.F. $= e^{\int \frac{2}{y}dy} = y^2$

$x(y^2) = \int 10y^4dy$

$y^2x = 2y^5 + C$

Q.32 (1)

$y' + y\phi' - \phi\phi' = 0$

$y' + \phi'(y - \phi) = 0$

$dy + \phi'(y - \phi)\,dx = 0$

Let $\phi = t \Rightarrow \phi'\,dx = dt$

$dy + (y - t)\,dt = 0$

$\frac{dy}{dt} + y = t$

I.F. $= e^t$

$ye^t = \int te^t\,dt$

$ye^t = te^t - e^t + C$

$y = t - 1 + ce^{-t}$

$y = \phi(x) - 1 + ce^{-\phi(x)}$

Q.33 (2)

$(1+y^2)\,dx + (x - e^{\tan^{-1}y})\,dy = 0$

$dx + \frac{(x - e^{\tan^{-1}y})dy}{(1+y^2)} = 0$

put $e^{\tan^{-1}y} = t \Rightarrow \frac{e^{\tan^{-1}y}}{(1+y^2)}\,dy = dt$

$\frac{dy}{(1+y^2)} = \frac{dt}{t}$

$dx + (x - t)\frac{dt}{t} = 0$

$tdx + xdt - tdt = 0$

$d(xt) - tdt = 0$

$xt = \frac{t^2}{2} + C$

$x\,e^{\tan^{-1}y} = \frac{1}{2}\,e^{2\tan^{-1}y} + C$

Q.34 (3)

$\frac{xdy}{x^2+y^2} = \frac{ydx}{x^2+y^2} - dx$

$\frac{xdy - ydx}{x^2+y^2} = -dx$

$\frac{\frac{xdy - ydx}{x^2}}{1+\left(\frac{y}{x}\right)^2} = -dx \Rightarrow \frac{d(y/x)}{1+(y/x)^2} = -dx$

$d\left(\tan^{-1}\frac{y}{x}\right) = -dx \Rightarrow \tan^{-1}\frac{y}{x} = -x + C$

$\frac{y}{x} = \tan(C - x) \Rightarrow y = x\tan(C - x)$

JEE-ADVANCED
OBJECTIVE QUESTIONS

Q.1 (A)

$P(x, y) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

$\frac{dP}{dx} = 0 \Rightarrow 2ax + 2hy + 2g = 0$ → $ax + hy + g = 0$

$\frac{dP}{dy} = 0 \Rightarrow 2by + 2hx + 2f = 0$ → $by + hx + f = 0$

$\Rightarrow h = g = f = 0$

conic : $ax^2 + by^2 + c = 0$

$\Rightarrow x^2 + \left(\frac{b}{a}\right)y^2 + \frac{c}{a} = 0$

order = 2

Q.2 (C)

$\frac{dy_1}{dx} + fy_1 = r$

$\frac{dy_2}{dx} + fy_2 = r$

Add $\frac{d}{dx}(y_1 + y_2) + f(y_1 + y_2) = 2r$

here $\frac{dy}{dx} + f(x)y = 2r$

Q.3 (A)

$y_1y_3 = 3y_2^2$

$\frac{y_3}{y_2} = 3\frac{y_2}{y_1} \Rightarrow \ell n\, y_2 = 3\ell n\, y_1 + \ell n\, c$

$y_2 = cy_1^3$

$\frac{y_2}{y_1^2} = cy_1$

$-\frac{1}{y_1} = cy + c_2$

$\frac{dx}{dy} = -cy - c_2$

$x = -\frac{cy^2}{2} - c_2y + c_3$

$\therefore\ x = A_1y^2 + A_2y + A_3$

Q.4 (B)

$\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 - y = 0$

$y = 0$, $x = 0$ — Two lines will satisfy above equation.

Q.5 (B)

$\frac{dy}{dx} = y - y^2 \Rightarrow \int\frac{dy}{y-y^2} = \int dx$

$\int\frac{1}{1-y} + \frac{1}{y}\,dy = x + c \Rightarrow \ell n\frac{y}{1-y} = x + c$

$\frac{y}{1-y} = ke^x \Rightarrow y = ke^x - kye^x \Rightarrow y = \frac{ke^x}{1+ke^x}$

$x = 0,\ y = 2;\ 2 = \frac{k}{1+k} \Rightarrow 2 + 2k = k$

$\Rightarrow k = -2,\ \ y = \dfrac{-2e^x}{1-2e^x} \Rightarrow y = \dfrac{-2}{e^{-x}-2}$

$\lim\limits_{x\to\infty}(y(x)) = \lim\limits_{x\to\infty}\dfrac{-2}{e^{-x}-2} = 1$

Q.6 (C)

$\dfrac{dy}{dx} = \dfrac{1}{2y} \Rightarrow y^2 = x + c$

$\because$ (4, 3) satisfies

$\Rightarrow 9 = 4 + c \Rightarrow c = 5$

$\therefore\ y^2 = x + 5$

Q.7 (D)

$\int_0^x t\,y(t)\,dt = x^2\,y(x)$

Using leibnitz

$xy = 2xy + x^2y'$

$y = 2y + xy'$

$xy' + y = 0$

$x\dfrac{dy}{dx} + y = 0 \Rightarrow \dfrac{dy}{y} + \dfrac{dx}{x} = 0$

$\ell n\, xy = C$

$xy = k$; $(2, 3) \Rightarrow k = 0$

$xy = 6$

Q.8 (A*)

$Y - y = m(x - x)$

$X_{int} = \dfrac{-y}{m} + x = ay$

$\dfrac{-y}{m} = ay - x \Rightarrow m = \dfrac{y}{x-ay} = \dfrac{dy}{dx}$

$\dfrac{dx}{dy} = \dfrac{x-ay}{y} \Rightarrow \dfrac{dx}{dy} - \dfrac{x}{y} = -a$

Q.9 (D)

$\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$

$\int\dfrac{dy}{\sqrt{1-y^2}} = \int\dfrac{-dx}{\sqrt{1-x^2}}$

$\Rightarrow -\sin^{-1} y = \sin^{-1} x + c$

$\sin^{-1} x + \sin^{-1} y = c$

Q.10 (A)

$\int_0^x t\,y(t)dt = x^2 + y(x)$

$xy(x) = 2x + y'(x)$; $y(a) = -a^2$

$x(y-2) = y'$

$xdx = \dfrac{y'}{y-2}$

$\dfrac{x^2}{2} = \ell n\,(y-2) + C$

$y - 2 = e^{\left(\frac{x^2}{2}-c\right)}$

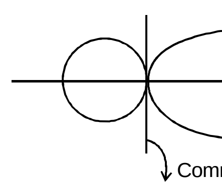
$\Rightarrow\ c = \dfrac{a^2}{2} - \ell n\,(-a^2-2)$

$y = 2 - (a^2+2)\, e^{\frac{x^2-a^2}{2}}$

Q.11 (A)

$Y - y = \dfrac{-dx}{dy}(X - x)$

$0 - y = \dfrac{-dx}{dy}(1 - x)$



$ydy = (1 - x)\,dx$

$\dfrac{y^2}{2} = x - \dfrac{x^2}{2} + C$

passing through (0, 0)

$\Rightarrow\ C = 0$

$x^2 + y^2 - 2x = 0$

$y^2 = 4x$

$x = 0 \rightarrow$ is common target

Q.12 (A)

$\dfrac{dy}{dx} = \sin(10x + 6y)$

Put $10x + 6y = t \quad \Rightarrow\ 10 + 6\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\dfrac{dt}{dx} - 10 = 6\sin t \quad \Rightarrow\ \dfrac{dt}{10+6\sin t} = dx$

$$\frac{dt}{10+6\left(\frac{2\tan\frac{t}{2}}{1+\tan^2\frac{t}{2}}\right)} = dx$$

$$\frac{\sec^2\frac{t}{2}dt}{10+10\tan^2\frac{t}{2}+12\tan\frac{t}{2}} = dx$$

put $\tan\frac{t}{2} = z \Rightarrow \sec^2\frac{t}{2}\,dt = 2dz$

$$\frac{2dz}{10(1+z^2)+12z} = dx$$

$$\frac{dz}{5z^2+6z+5} = dx \Rightarrow \frac{dz}{\left(z+\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2} = 5\,dx$$

$$\frac{5}{4}\tan^{-1}\frac{z+\frac{3}{5}}{\frac{4}{5}} = 5x + 5k$$

$$\tan^{-1}\frac{5z+3}{4} = 4x + C$$

$(0, 0) \Rightarrow C = \tan^{-1}\frac{3}{4}$

$$\tan^{-1}\frac{5z+3}{4} - \tan^{-1}\frac{3}{4} = 4x$$

$$\frac{\frac{5z+3}{4}-\frac{3}{4}}{1+\left(\frac{5z+3}{4}\right)\frac{3}{4}} = \tan 4x \Rightarrow \frac{20z}{25+15z} = \tan 4x$$

$4z = (5 + 3z)\tan 4x$

$z(4 - 3\tan 4x) = 5\tan 4x$

$$z = \frac{5\tan 4x}{4-3\tan 4x} \Rightarrow \tan\frac{t}{2} = \left(\frac{5\tan 4x}{4-3\tan 4x}\right)$$

$$5x + 3y = \tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right)$$

$$y = \frac{1}{3}\tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right) - \frac{5x}{3}$$

Q.13 (A)

$$\frac{dy}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$$

Put $y = tx \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$$t + x\frac{dt}{dx} = t - \cos^2(t) \Rightarrow x\frac{dt}{dx} = -\cos^2 t$$

$$\sec^2 t\,dt + \frac{dx}{x} = 0$$

$\tan t + \ell n\, x = C \Rightarrow \tan t = C - \ell n\, x$

$$\tan\frac{y}{x} = C - \ell n\, x \quad \left(1, \frac{\pi}{4}\right)$$

$C = 1$

$$\tan\frac{y}{x} = 1 - \ell n\, x \Rightarrow \tan\frac{y}{x} = \ell n\frac{e}{x}$$

$$y = x\tan^{-1}\left(\ell n\frac{e}{x}\right)$$

Q.14 (A)

$$\int_0^1 f(tx)\,dt = n\,f(x)$$

let $tx = z \Rightarrow dt = dz/x$

$$\int_0^x f(z).\frac{dz}{x} = n\,f(x)$$

$$\int_0^x f(z)\,dz = nx\,f(x)$$ use leibnitz to differentiate

$f(x) = n\,f(x) + n\,x\,f'(x)$

$nx\,f'(x) = (1 - n)\,f(x)$

$$\frac{f'(x)}{f(x)} = \left(\frac{1-n}{n}\right)\frac{1}{n}$$

$$\ell n\, f(x) = \left(\frac{1-n}{n}\right)\ell n\, x + \ell n\, c$$

$$\Rightarrow f(x) = k\,.\,x^{\left(\frac{1-n}{n}\right)}$$

Q.15 (B)

(i) $\frac{dy_1}{dx} + f(x)\,y_1 = 0 \Rightarrow f(x) = \frac{-1}{y_1}\frac{dy_1}{dx}$

(ii) $\frac{dy}{dx} - \frac{1}{y_1}\frac{dy_1}{dx}\,.\,y = r(x)$

$$e^{-\int\frac{1}{y_1}\frac{dy_1}{dx}dx} = e^{-\int\frac{dy_1}{y_1}} = \frac{1}{y_1}$$

$$\frac{d}{dx}\left(\frac{y}{y_1}\right) = \frac{r(x)}{y_1} \Rightarrow \frac{y}{y_1} = \int \frac{r(x)dx}{y_1}$$

$$y = y_1\int \frac{r(x)dx}{y_1}$$

Q.16 (B)

$e^x (x + 1)\, dx + (ye^y - xe^x)\, dy = 0$

Put $xe^x = t \Rightarrow e^x (x + 1)\, dx = dt$

$dt + (ye^y - t)\, dy = 0$

$$\frac{dy}{dt} = \frac{1}{t - ye^y} \Rightarrow \frac{dt}{dy} = t - ey^y$$

$$\frac{dt}{dy} - t = -ye^y$$

I.F. $= e^{-y}$

$$(xe^x)(e^{-y}) = \frac{-y^2}{2} + C; \quad (0, 0) \Rightarrow C = 0$$

$$(xe^x)(e^{-y}) = \frac{-y^2}{2}$$

$2xe^x + y^2e^y = 0$

Q.17 (B)

$$y^5x + y - x\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} - y - y^5 x = 0$$

$$\frac{dy}{dx} - \frac{y}{x} = y^5 \qquad \text{(LDE)}$$

$$\frac{1}{y^5}\frac{dy}{dx} - \frac{1}{y^4x} = 1$$

$$\frac{1}{y^4} = t \Rightarrow \frac{1}{y^5}\frac{dy}{dx} = -\frac{1}{4}\frac{dt}{dx}$$

$$-\frac{1}{4}\frac{dt}{dx} - \frac{t}{x} = 1$$

$$\frac{dt}{dx} + \frac{4t}{x} = -4 \qquad \text{(LDE)}$$

$$\text{I.F.} = e^{\int \frac{4}{x}dx} = x^4$$

$$x^4 . t = -\frac{4x^5}{5} + C$$

$$\frac{x^4}{y^4} = -\frac{4x^5}{5} + C$$

$$\frac{x^5}{5} + \frac{1}{4}\left(\frac{x}{y}\right)^4 = k$$

Q.18 (A)

$(y + 3x^2y^2e^{x^3})dx = x\, dy$

$$\Rightarrow \frac{dy}{dx} = \frac{y + 3x^2y^2e^{x^3}}{x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = y^2(3xe^{x^3})$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{xy} = 3xe^{x^3}$$

$$\text{put } -\frac{1}{y} = t \Rightarrow \frac{dt}{dx} + \frac{t}{x} = 3xe^{x^3}$$

$$\text{I.F.} = e^{\int \frac{dx}{x}} = x$$

$$\frac{d}{dx}(t\,x) = 3x^2e^{x^3} \Rightarrow tx = \int 3x^2e^{x^3}dx$$

$$\Rightarrow \frac{-x}{y} = e^{x^3} + c$$

Q.19 (A)

$$x^2\frac{dy}{dx} . \cos\frac{1}{x} - y\sin\frac{1}{x} = -1$$

$$\frac{dy}{dx} - \frac{y}{x^2}\tan\frac{1}{x} = \frac{-1}{x^2}\sec\frac{1}{x}$$

$$\text{I.F.} = e^{-\int \frac{1}{x^2}\tan\frac{1}{x}dx} = \sec\frac{1}{x}$$

$$y\left(\sec\frac{1}{x}\right) = \int \frac{-1}{x^2}\sec^2\frac{1}{x}\, dx$$

$$y\left(\sec\frac{1}{x}\right) = \tan\frac{1}{x} + C$$

$C = -1$

$$y\left(\sec\frac{1}{x}\right) = \tan\frac{1}{x} - 1$$

$$y = \sin\frac{1}{x} - \cos\frac{1}{x}$$

Q.20 (C)

$$2xy\frac{dy}{dx} = (x^2 + y^2 + 1)$$

Put $x^2 + y^2 + 1 = t$

$$2x + 2y\frac{dy}{dx} = \frac{dt}{dx}$$

$$2x^2 + 2xy\frac{dy}{dx} = x\frac{dt}{dx}$$

$$x\frac{dt}{dx} - 2x^2 = t \Rightarrow x\frac{dt}{dx} = t + 2x^2$$

$$\frac{dt}{dx} - \frac{t}{x} = 2x$$

$$\frac{t}{x} = 2x + C \Rightarrow t = 2x^2 + Cx$$

$x^2 + y^2 + 1 = 2x^2 + Cx$

$x^2 - y^2 + Cx - 1\ x = 0$

Q.21 (D)

$$y = \frac{x}{\ell n(cx)}$$

$$y' = \frac{\ell n(x).1 - \frac{x}{cx}}{(\ell ncx)^2} = \frac{(\ell ncx) - \frac{1}{c}}{(\ell ncx)^2}$$

$$\frac{dy}{dx} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$$

$$\frac{\ell ncx - \frac{1}{c}}{(\ell ncx)^2} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$$

$$\frac{\frac{x}{y} - \frac{1}{c}}{\left(\frac{x}{y}\right)^2} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$$

$$\frac{y}{x} - \frac{1}{c\left(\frac{x}{y}\right)^2} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$$

$$\phi\left(\frac{x}{y}\right) = -\frac{1}{c}\left(\frac{y^2}{x^2}\right)$$

Q.22 (A)

$(x^3\cos y\ \sin^2 y - 2y\ \sin x)\ dy - (y^2\ \cos x - x^2\ \sin^3 y)\ dx = 0$

$$\left(\frac{x^3}{3}d\sin^3 y - \sin x\, dy^2\right) + \sin^3 y d\left(\frac{x^3}{3}\right) - y^2 d\sin x = 0$$

$$\frac{x^3}{3}\ d\sin^3 y + \sin^3 y d\left(\frac{x^3}{3}\right) - (\sin x\ dy^2 + y^2\ d\sin x)$$

$$d\left(\frac{x^3}{3}\sin^3 y\right) - d(y^2\ \sin x) = 0$$

$$\frac{x^3}{3}\sin^3 y - y^2 \sin x = c$$

Q.23 (A)

$x^2\ dy - y^2 dx + x^2y^2\ dy - xy^3\ dy = 0$

$$\frac{1}{y^2}\ dy - \frac{1}{x^2}\ dx + dy - \frac{y}{x}dy = 0$$

$$d\left(\frac{1}{x} - \frac{1}{y}\right) = \left(\frac{y-x}{xy}\right)y\ dy$$

$$\frac{d\left(\frac{1}{x} - \frac{1}{y}\right)}{\left(\frac{1}{x} - \frac{1}{y}\right)} = ydy$$

$$\Rightarrow \ell n\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{y^2}{2} + c$$

$$\ell n\left|\frac{y-x}{yx}\right| = \frac{y^2}{2} + c$$

$$\ell n\left|\frac{y-x}{yx}\right| = \frac{y^2}{2} + c$$

Q.24 (C)

$$\frac{dy}{dt} = k\sqrt{y} \Rightarrow \int_0^y \frac{dy}{\sqrt{y}} = \int_0^t k\,dt$$

$$2\sqrt{y} = kt \Rightarrow t = \frac{2\sqrt{y}}{k}$$

$t = 2 \times 2 \times 15 = 60$ min

JEE-ADVANCED
MCQ/COMPREHENSION/COLUMN MATCHING

Q.1 (C, D)

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \sin y + x^2 = 0$$

Order = 1, Degree = 1

Q.2 (A, D)

$$\left(\frac{dy}{dx}\right)^2 - \left(\frac{dy}{dx}\right)(e^x + e^{-x}) + 1 = 0$$

$$\frac{dy}{dx} = \frac{(e^x + e^{-x}) \pm \sqrt{(e^x + e^{-x})^2 - 4}}{2}$$

$$\frac{dy}{dx} = \frac{(e^x + e^{-x}) \pm (e^x - e^{-x})}{2}$$

+ve $\frac{dy}{dx} = e^x \Rightarrow y = e^x + C_1$

–ve $\frac{dy}{dx} = e^{-x} \Rightarrow y = -e^{-x} + C_2$

Q.3 (A, C, D)

$y = (A + Bx)\, e^{3x}$

$y' = 3(A + Bx)\, e^{3x} + Be^{3x} = e^{3x}\,(3A + B + 3Bx)$

$y' = 3y + Be^{3x}$

$y'' = 3y' + 3Be^{3x}$

By adding something and subtracting it will convert into

$y'' - 3y' = 3y + 3(B + 3A + 3Bx)\, e^{3x}$

$- 12\,(A + Bx)\, e^{3x}$

$y'' - 3y' = 3y + 3y' - 12\, y$

$y'' - 6y' + 9y = 0$

$m = -6,\ n = 9$

Q.4 (B, C)

(A) $\frac{dy}{dx} = -c_1 \sin x + \sqrt{c_2} \cos x$

$\frac{d^2y}{dx^2} = -c_1 \cos x - \sqrt{c_2} \sin x = 2 - y$

$\frac{d^2y}{dx^2} + y + 2 = 0$

(B) $\frac{dy}{dx} = \cos x \,\frac{\sec^2 x/2}{2\tan\frac{x}{2}} - \sin x\, \ell n\left(\tan\frac{x}{2}\right)$

$\frac{dy}{dx} = \cot x - \sin x\, \ell n\left(\tan \frac{x}{2}\right)$

$\frac{d^2y}{dx^2}$

$= -\text{cosec}^2 x - \left(\sin x.\frac{1}{\sin x} + \cos x\, \ell n\left(\tan\frac{x}{2}\right)\right)$

$\frac{d^2y}{dx^2} = -\cot^2 x - 2 - \cos x\, \ell n\left(\tan\frac{x}{2}\right)$

$\frac{d^2y}{dx^2} + y + \cot^2 x = 0$

(C) $\frac{dy}{dx} = -c_1 \sin x + c_2 \cos x + \frac{d}{dx}\left(\cos x\, \ell n\left(\tan \frac{x}{2}\right)\right)$

$\frac{d^2y}{dx^2} = -(c_1 \cos x + c_2 \sin x) + \frac{d^2}{dx^2}\left(\cos x\, \ell n\left(\tan \frac{x}{2}\right)\right)$

$\frac{d^2y}{dx^2} = -c_1 \cos x - c_2 \sin x - \cot^2 x - 2 - \cos x\, \ell n\left(\tan \frac{x}{2}\right)$

$\frac{d^2y}{dx^2} + y + \cot^2 x = 0$

Q.5 (A)

$y = e^{-x} \cos x$

$y_1 = -e^{-x} \cos x - e^{-x} \sin x = -e^{-x} (\sin x + \cos x)$

$= -\sqrt{2}\; e^{-x} (\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4})$

$y_1 = -\sqrt{2}\; e^{-x} \cos\left(x - \frac{\pi}{4}\right)$

$y_2 = +\sqrt{2}\; e^{-x} \cos\left(x - \frac{\pi}{4}\right) + \sqrt{2}\; e^{-x} \sin\left(x - \frac{\pi}{4}\right)$

$= \sqrt{2}\; e^{-x} \left(\cos\left(x - \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)\right)$

$= \sqrt{2}\,.\,\sqrt{2}\; e^{-x} \cos\left(x - \frac{\pi}{4} - \frac{\pi}{4}\right)$

similarly

$= \left(\sqrt{2}\right)^2 e^{-x} \cos\left(x - \frac{2\pi}{4}\right)$

$y_3 = -\left(\sqrt{2}\right)^3 e^{-x} \cos\left(x - \frac{3\pi}{4}\right)$

$y_4 = +\left(\sqrt{2}\right)^4 e^{-x} \cos\left(x - \frac{4\pi}{4}\right)$

$= -\left(\sqrt{2}\right)^4 e^{-x} \cos x$

$y_4 + 4y = 0$

Similarly

$y_8 = \left(\sqrt{2}\right)^8 e^{-x} \cos\left(x - \frac{8\pi}{4}\right) = \left(\sqrt{2}\right)^8 e^{-x} \cos\, (x - 2\pi)$

$= 16\, e^{-x} \cos x$

$y_8 - 16y = 0$

Q.6 (A, C, D)

$x^2 y_1^2 + xy\, y_1 - 6y^2 = 0$

It is quadratic equation in y_1

$y_1 = \frac{-xy \pm \sqrt{x^2y^2 + 24y^2x^2}}{2x^2} = \frac{-xy \pm 5xy}{2x^2}$

$y_1 = -\frac{3y}{x}$ | $y_1 = \frac{2y}{x}$

$\frac{dy}{dx} = \frac{-3y}{x}$ | $\frac{dy}{dx} = \frac{2y}{x}$

$-\frac{dy}{y} = 3\frac{dx}{x}$ | $\frac{dy}{dx} = \frac{2y}{x}$

$-\ell n\ y = 3\ \ell nx + \ell nc$ | $\ln y = 2\ \ell nx + \ell nc$

$x^3y = C$

$y = cx^2$

Option (C)

$\frac{1}{2}\log y = c + \log x$

$\log y = \ell n\ c_1 + \log x^2$

$y = C_1x^2$

Q.7 (A, D)

$$\frac{dy}{1+y^2} + \frac{dx}{\sqrt{1-x^2}} = 0$$

$\Rightarrow \tan^{-1} y + \sin^{-1} x + c = 0$

$\Rightarrow \cot^{-1}\frac{1}{y} + \cos^{-1}\sqrt{1-x^2} + c = 0$

Q.8 (C, D)

$y^2 + 4y'y + (y')^2 = 0$

$$y' = \frac{-4y \pm \sqrt{16y^2 - 4y^2}}{2}$$

$$\frac{dy}{dx} = \frac{-4y \pm 2\sqrt{3}y}{2}$$

+ve $\frac{dy}{y} = (-2 + \sqrt{3})\,dx$

$\ell ny = (-2 + \sqrt{3})x + \ell nC$

$y = k\,e^{(-2+\sqrt{3})x}$

–ve $y = k\,e^{(-2-\sqrt{3})x}$

Q.9 (A, B, C)

Its not asking homogeneous diff. equation

It is asking homogeneous function.

So (A) ✓ (B)✓ (C) ✓(D) ×

Q.10 (A, B)

$$\frac{dy}{dx} = \frac{-(x-y) \pm \sqrt{(x-y)^2 + 4xy}}{2y}$$

+ve $\frac{dy}{dx} = \frac{-(x-y)+(x+y)}{2y}$

$\frac{dy}{dx} = 1 \Rightarrow y = x + C$

passes through (3, 4)

$y - x = 1$

–ve $\frac{dy}{dx} = \frac{-(x-y)-(x+y)}{2y}$

$\frac{dy}{dx} = \frac{-x}{y}$

$\Rightarrow ydy + xdx = 0$

$y^2 + x^2 = 25$

Q.11 (C)

Given DE can be written as

$$\frac{dy}{dx} - \left(1 + \frac{f'(x)}{f(x)}\right) y = f(x)$$

Which is L.D.E.

I.F. $= e^{-x-\ell n\ f(x)} = \frac{e^{-x}}{f(x)}$

General solution

$$y\frac{e^{-x}}{f(x)} = \int f(x)\frac{e^{-x}}{f(x)}\,dx + c = -e^{-x} + c$$

$\Rightarrow y = -f(x) + ce^x f(x)$

Q.12 (A, D)

$\frac{dx}{dy} = x + y + 1$

$\Rightarrow \frac{dx}{dy} - x - y - 1 = 0$ I.F. $= e^{-\int dy} = e^{-y}$

$\Rightarrow e^{-y}\frac{dx}{dy} - xe^{-y} - ye^{-y} - e^{-y} = 0$

$\Rightarrow \int d(xe^{-y}) = \int (e^{-y} + ye^{-y})\,dy$

$\Rightarrow xe^{-y} = -e^{-y} - ye^{-y} + \int e^{-y}\,dy$

$\Rightarrow xe^{-y} = -e^{-y} - ye^{-y} - e^{-y} + c$

$\Rightarrow x = -1 - y - 1 + ce^y$

$\Rightarrow x + y + 2 = ce^y$

Q.13 (A,B,D)

$\frac{dy}{dx} + y\cos x = \cos x$

I.F. $= e^{\int \cos x\,dx} = e^{\sin x}$

$y\,e^{\sin x} = \int e^{\sin x}\cos x\,dx = e^{\sin x} + C$

$y = 1 + ce^{-\sin x}$; C = 0 as (0, 1)

$y = 1$

Q.14 (A, B, C)

$$\left(\frac{dy}{dx}\right) = \pm\sqrt{\frac{a}{x}}$$

orthogonal $-\frac{dx}{dy} = \pm\sqrt{\frac{a}{x}}$

$\Rightarrow \int \sqrt{x}\,dx = \int \pm\sqrt{a}\,dy$

$\Rightarrow \frac{x^{3/2}}{3/2} = \pm\sqrt{a}\,y + c$

it can also be written as

$y + k = \pm\frac{2}{3}\frac{x^{3/2}}{3\sqrt{a}}$

Q.15 (A, B, C, D)

$\left(\frac{dy}{dx}\right)^2 = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \pm\frac{2}{\sqrt{x}}$

Orthogonal Trajectory

$\frac{dy}{dx} = \mp\frac{\sqrt{x}}{2}$

By integrating

$y + C = \pm\frac{x^{3/2}}{3}$ or $9(y + C)^2 = x^3$

Comprehension # 1 (Q.no. 16 to 18)

Q.16 (D)

Q.17 (C)

Q.18 (A)

(16 to 18)

$x^2 + y^2 - ax = 0$

$a = \frac{x^2 + y^2}{x}$

$0 = \frac{x(2x + 2yy') - (x^2 + y^2)}{x^2}$

$\Rightarrow x^2 + 2xy\,y' - y^2 = 0$

orthogonal curve

$x^2 - \frac{2xy}{y'} - y^2 = 0$

$\Rightarrow y' = \frac{2xy}{x^2 - y^2}$ ... (1)

Put $y = vx$

$v + x\frac{dv}{dx} = \frac{2v}{1 - v^2}$

$\Rightarrow x\frac{dv}{dx} = \frac{2v}{1 - v^2} - v = \frac{v + v^3}{1 - v^2}$

$\Rightarrow \frac{1 - v^2}{v + v^3}dv = \frac{dx}{x}$

$\Rightarrow \frac{(1 + v^2) - 2v^2}{v(1 + v^2)}dv = \frac{dx}{x}$

$\left(\frac{1}{v} - \frac{2v}{1 + v^2}\right)dv = \frac{dx}{x}$

$\Rightarrow \ln v - \ln(1 + v^2) = \ln x + \ln c$

$\Rightarrow \ln\left(\frac{v}{1 + v^2}\right) = \ln(cx)$

$\Rightarrow \frac{v}{1 + v^2} = cx \Rightarrow \frac{xy}{x^2 + y^2} = cx$

$\Rightarrow x^2 + y^2 - \frac{y}{c} = 0.$

This passes through (1, 1)

$\therefore\ c = \frac{1}{2}$

$\therefore\ C : x^2 + y^2 - 2y = 0$

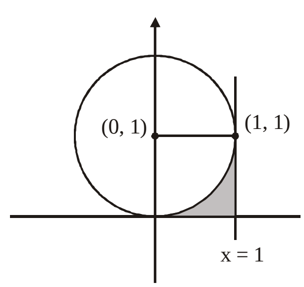
(i) Since $x^2 + y^2 + 2x + 4y + k = 0$ and $x^2 + y^2 - 2y = 0$ are orthogonal

$\therefore\ 2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$\therefore\ k = -4$

(ii) Required area

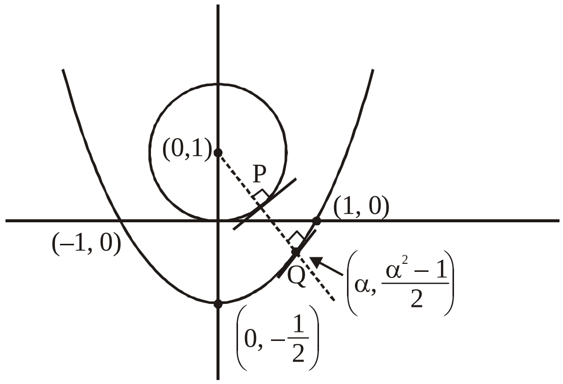
$= 1 - \frac{\pi}{4}$



(iii) Let $Q\left(\alpha, \frac{\alpha^2 - 1}{2}\right)$

Slope of tangent at P and Q are same

m_{PQ} = slope of normal at Q



$$\frac{-1}{\alpha} = \frac{\frac{\alpha^2-1}{2}-1}{\alpha-0} \Rightarrow \alpha^3 - \alpha = 0 \Rightarrow \alpha = 0, \pm 1$$

$\therefore$ Q(1, 0)

PQ = $\sqrt{2}$ – 1.]

Comprehension # 2(Q.no. 19 to 21)

Q.19 (B)

$2x^3dx + 2y^3dy - (xy^2\,dx + x^2y\,dy) = 0$

$$d\left(\frac{x^4}{2}\right) + d\left(\frac{y^4}{2}\right) - \frac{1}{2}\,d\,(x^2y^2) = 0$$

$\Rightarrow d\,(x^4 + y^4 - x^2y^2) = 0 \Rightarrow x^4 + y^4 - x^2y^2 = c$

Q.20 (A)

$$\frac{xdy - ydx}{x^2+y^2} + dx = 0 \Rightarrow \frac{\frac{xdy-ydx}{x^2}}{1+\left(\frac{y}{x}\right)^2} + dx = 0$$

$$\Rightarrow \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} + dx = 0$$

$$\Rightarrow d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + dx = 0 \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) + x = c$$

Q.21 (A)

$e^ydx + xe^ydy - 2ydy = 0$

$d\,(xe^y) - d(y^2) = 0$

Solution is $xe^y - y^2 = c$

Comprehension # 3 (Q.no. 22 to 24)

Q.22 (A)

Q.23 (B)

Q.24 (C)

(22 to 24)

$D^2 - 2D + 1 = 0$

$(D - 1)^2 = 0$

Two real and equal roots $\alpha_1 = 1$

$y = (C_1 + C_2x)e^x$

$D^2 + a^2 = 0$

$D = \pm ai$ one pair of imaginary roots $\alpha \pm i\beta$

$\alpha = 0$, $\beta = a$

$y = (C_1 \cos \beta x + C_2 \sin \beta x)e^{\alpha x}$

$y = (C_1 \cos ax + C_2 \sin ax)$

$D^3 - 6D^2 + 11D - 6 = 0$

$\Rightarrow D^3 - D^2 - 5D^2 + 5D + 6D - 6 = 0$

$\Rightarrow (D - 1)(D^2 - 5D + 6) = 0$

$\Rightarrow (D - 1)(D - 2)(D - 3) = 0$

three real and different roots $\alpha_1, \alpha_2, \alpha_3$

$\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3$

$y = C_1e^x + C_2e^{2x} + c_3e^{3x}$

Q.25 (A) – (r); (B) – (p); (C) – (s); (D) – (q)

(A) $\frac{ydx - xdy}{y^2} = dx + \frac{dy}{y^2} \Rightarrow d\left(\frac{x}{y}\right) = dx + \frac{dy}{y^2} \Rightarrow$

$\frac{x}{y} = x - \frac{1}{y} + k$

$\Rightarrow x = xy - 1 + ky \quad \Rightarrow (x + 1)(1 - y) = cy$

(B) $(2x - 10y^3)\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{10y^3 - 2x}$

$$\Rightarrow \frac{dx}{dy} = \frac{10y^3 - 2x}{y}$$

$$\frac{dx}{dy} = 10y^2 - 2\frac{x}{y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{2}{y}x = 10y^2$$

$$\Rightarrow xy^2 = 10\frac{y^5}{5} + c$$

$\Rightarrow xy^2 = 2y^5 + c$

(C) $\sec^2 y\,\frac{dy}{dx} + \tan y = 1$ put $\tan y = t$

$$\sec^2 y\,\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dt}{dx} = 1 - t$$

$\Rightarrow \ell n\,(1 - t) = -c\,x$

$\Rightarrow 1 - t = e^{-cx}$

$\Rightarrow t = 1 - e^{-cx} \Rightarrow \tan y = 1 + ce^{-x}$

(D) $\sin y \dfrac{dy}{dx} = \cos y\,(1 - x\cos y)$

put $\cos y = t$ $\qquad -\sin y \dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\Rightarrow -\dfrac{dt}{dx} = t(1 - tx) \Rightarrow \sec y = x + 1 + ce^{x}$

Q.26 (A) – (p,s); (B) → (q); (C) → (s); (D) → (r)

(A) RV = $\sqrt{x^2 + y^2} = \dfrac{y}{m}\sqrt{1+m^2}$

$\Rightarrow m^2(x^2 + y^2) = y^2(1 + m^2)$

$\Rightarrow m^2 = \dfrac{y^2}{x^2} \Rightarrow m = \pm\dfrac{y}{x}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x}$ or $\dfrac{dy}{dx} = \dfrac{-y}{x}$

$\Rightarrow \int\dfrac{dy}{y} = \int\dfrac{dx}{x}$ or $\int\dfrac{dy}{y} = \int\dfrac{-dx}{x}$

$\Rightarrow y = x\,c_1$ or $y = \dfrac{C_2}{x} \Rightarrow xy = C_2$

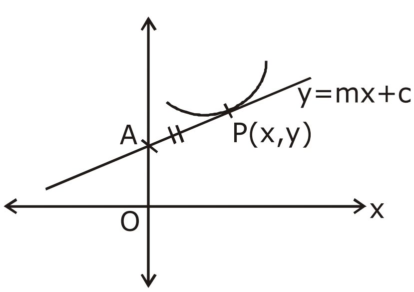
Straight line Hyperbola

(B) A(0, y – mx)

OA = OP

$\Rightarrow (y - mx)^2 = x^2 + m^2x^2$

$\Rightarrow m = \dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy} = \dfrac{(y/x)^2 - 1}{2(y/x)}$



Put y = vx

$\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\Rightarrow \int\dfrac{2v}{V^2+1}\dfrac{dv}{dx} = \dfrac{-dx}{x}$

$\Rightarrow \ln|v^2 + 1| = -\ln|x| + \ln C$

$\Rightarrow \dfrac{y^2}{x^2} + 1 = \dfrac{c}{x} \Rightarrow y^2 + x^2 = cx$ circle

(C) RV = $\sqrt{x^2 + y^2} = y\sqrt{1+m^2}$

$\Rightarrow x^2 + y^2 = y^2(1 + m^2)$

$\Rightarrow x^2 + y^2 = y^2 + m^2y^2$

$\Rightarrow m^2 = \dfrac{y^2}{x^2} \Rightarrow \dfrac{dy}{dx} = \pm\dfrac{y}{x}$

$\Rightarrow \int\dfrac{dy}{y} = \int\dfrac{dx}{x^2}$ or $\int\dfrac{dy}{y} = \int\dfrac{dx}{x}$

$\Rightarrow \ln y = \ln x + \ln c$ or $\ln y = -\ln x + \ln c$

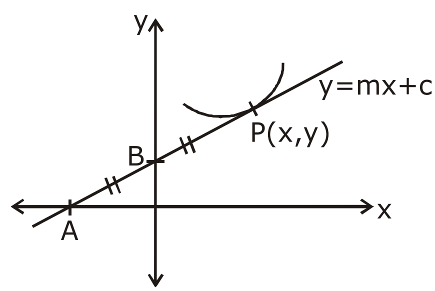
$y = x\,c$ or $y = \dfrac{c}{x} \Rightarrow xy = c$

Straight line Hyperbola

(D) B(0, c) $\qquad A\left(\dfrac{-c}{m}, 0\right)$

B(0, y – mx) $\qquad A\left(\dfrac{mx - y}{m}, 0\right)$

$\dfrac{mx - y}{0, m} + \dfrac{x}{B} = 0$



$mx - y + mx = 0$

$\Rightarrow 2mn = y \qquad \Rightarrow 2m = \dfrac{y}{x}$

$\Rightarrow \dfrac{2dy}{dn} = \dfrac{y}{n} \qquad \Rightarrow \int 2\dfrac{dy}{y} = \int\dfrac{dn}{x}$

$\Rightarrow 2\ln y = \ln x + \ln c$

$\Rightarrow y^2 = x\,c$ Parabola

Q.27 (A) – (q), (B) – (r), (C) – (s), (D) – (p)

(A) $x\,dy = y\,dx + y^2\,dy$

$\Rightarrow \dfrac{x\,dy - y\,dx}{y^2} = dy \Rightarrow -d\left(\dfrac{x}{y}\right) = dy$

$-\dfrac{x}{y} = y + c$ put x = 1 $\qquad$ y = 1 ⇒ c = –2

$-\dfrac{x}{y} = y - 2 \Rightarrow \dfrac{x}{3} = -5$

(B) $\dfrac{dy}{dt} - \dfrac{t}{t+1}y = \dfrac{1}{t+1}$

I.F = $e^{-\int \frac{t+1-1}{t+1} dt} = e^{-t + \ell n(t+1)}$ = (t + 1) e^{-t}

solution is (t + 1)e^{-t} y = $-e^{-t}$ + c

put t = 0 and y = – 1 $\Rightarrow$ c = 0

$\therefore$ $2e^{-1}$ y = – e^{-1}

$y = -\frac{1}{2}$

(C) $(x^2 + y^2)\, dy = xy\, dx$

$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ put y = vx $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow$ $\ell n\, v - \frac{1}{2v^2} = -\ell n\, x + c$

$\therefore$ $c = -\frac{1}{2}$

$\therefore$ $\ell n \frac{y}{x} - \frac{1}{2}\frac{x^2}{y^2} = -\ell n\, x - \frac{1}{2}$ put y = e

$\therefore$ $x = \sqrt{3}\, e$

(D) $\frac{dy}{dx} + 2\frac{y}{x} = 0$

$x^2 y = C$ put x = 1, y = 1 and we get C = 1

put x = 2 $\Rightarrow y = \frac{1}{4}$

NUMERICAL VALUE BASED

Q.1 [4]

$\frac{dy}{dx} = y + \int_0^1 y\, dx$

$\Rightarrow \frac{dy}{dx} = y + a$ let $\int_0^1 y\, dx = a$

$\Rightarrow \frac{dy}{dx} - y - a = 0$

I.F. = $e^{-\int 1.dx} = e^{-x}$

Now $e^{-x}\frac{dy}{dx} - ye^{-x} - ae^{-x} = 0$

$\Rightarrow ye^{-x} + ae^{-x} + c = 0$

$\Rightarrow y + a + ce^{x} = 0$

$a = \int_0^1 y\, dx = -\int_0^1 (a + ce^x)\, dx = -a(1-0) - c\,(e^x)\Big|_0^1$

$= -a - c(e-1)$

$\Rightarrow a = \frac{(1-e)c}{2}$

$y + \left(\frac{(1-e) + 2e^x}{2}\right)c = 0$

y = 1; x = 0 $\Rightarrow c = \frac{2}{e-3}$

$y(x) = \left(\frac{2e^x - e + 1}{3-e}\right)$

$y\left(\ell n\frac{11-3e}{2}\right) = \frac{11-3e-e+1}{3-e} = \frac{4[3-e]}{3-e} = 4$

Q.2 [35]

$c(y+c)^2 = x^3$ ..... (i)

dff^n, w.r.t x

$2c(y+c)\, y' = 3x^2$ .... (ii)

devide equation (i) by square of equation (ii)

$\frac{c(y+c)^2}{4c^2(y+c)^2(y')^2} = \frac{x^3}{9x^4}$

$c = \frac{9x}{4(y')^2}$

by putting c in equation (ii) we get

$12y(y')^2 + 27x = 8x(y')^3$

Q.3 [4]

$y\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y\frac{dy}{dx} - x = 0$

$y\frac{dy}{dx}\left(\frac{dy}{dx} - 1\right) + x\left(\frac{dy}{dx} - 1\right) = 0$

$\left(y\frac{dy}{dx} + x\right)\left(\frac{dy}{dx} - 1\right) = 0$

$\therefore$ either ydy + xdx = 0

or dy – dx = 0

since the curves pass through the point (3, 4)

$\therefore$ $x^2 + y^2 = 25$ or $x - y + 1 = 0$

$\Rightarrow$ $2x - 2y + 2 = 0$ $\Rightarrow$ A = 2 & B = 2

$\Rightarrow$ A – B = 4

Q.4 [25]

Equ. of tangent

Y – y = m(X - x)

$\left|\frac{y - mx}{\sqrt{1+m^2}}\right| = x$

$\Rightarrow (y - mx)^2 = x^2(1 + m^2)$

$\Rightarrow y^2 - 2mxy = x^2 \qquad \Rightarrow \frac{y^2 - x^2}{2xy} = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ Homogeneous equation

which gives $x^2 + y^2 - cx = 0$

$x^2 + y^2 = cx$

curve passes through (1, 3)

$\Rightarrow \quad c = 10$

equation of curve is $x^2 + y^2 - 10x = 0$

equation of tangent at (1, 3) is $x.1 + 3y - 5(x + 1) = 0$

$4x - 3y + 5 = 0$

$\Rightarrow \quad a^2 + b^2 = 4^2 + (-3)^2 = 25$

Q.5 [1]

$\frac{dy}{dx} = \frac{x^2 - y^2}{x^2 + y^2}$ let the line from origin be $y = mx$

$\frac{dy}{dx} = \frac{1 - m^2}{1 + m^2}$ which is constant and independent of x, y

Hence $f'(x_1) = g'(x_1)$

Q.6 [16]

$y(x + y^3)\,dx = x(y^3 - x)\,dy$

$\Rightarrow -\frac{y}{x} d\left(\frac{y}{x}\right) + \frac{d(xy)}{(xy)^2} = 0$

$\Rightarrow y^3 + 2x + 2cx^2y = 0$

It passes through (4, –2)

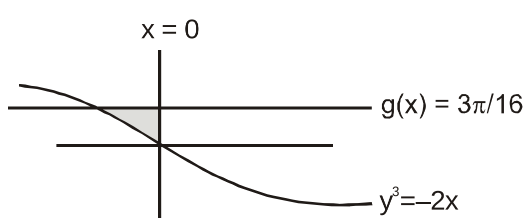
$-8 + 8 + 2c(16)(-2) = 0 \Rightarrow c = 0$

$\therefore \; y^3 = -2x$

$g'(x) = 2\sin x\cos x\sin^{-1}\sin x + 2\cos x(-\sin x)\cos^{-1}(\cos x) = 0$

so g(x) is constant

$g(\pi/4) = g(x) = \int_{1/8}^{1/2} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})\,dt$



$\Rightarrow g(x) = \frac{\pi}{2}\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{3\pi}{16}$

$A = \left| -\frac{1}{2}\int_0^{\frac{3\pi}{16}} y^3 dy \right|$; $\qquad A = \frac{1}{2x4}\left(\frac{3\pi}{16}\right)^4$

Q.7 [2]

$\frac{dy}{dx} = \frac{1}{x\cos y + 2\sin y\cos y}$

$\therefore \quad \frac{dx}{dy} = x\cos y + 2\sin y\cos y$

$\frac{dx}{dy} + (-\cos y)\,x = 2\sin y\cos y$

$\therefore \quad$ I.F. $= e^{-\int \cos y\,dy} = e^{-\sin y}$

$\therefore \quad$ The solution is

$x.\,e^{-\sin y} = 2\int e^{-\sin y}\,.\,\sin y\cos y\,dy$

$= -2\sin y\,e^{-\sin y} - 2\int(-e^{-\sin y})\cos y\,dx$

$= -2\sin y\,e^{-\sin y} + 2\int e^{-\sin y}\cos y\,dy$

$= -2\sin y\,e^{-\sin y} - 2e^{-\sin y} + c$

i.e.

$x = -2\sin y - 2 + c\,e^{\sin y} = ce^{\sin y} - 2(1 + \sin y)$

$\therefore \quad k = 2$

Q.8 [8]

$\cos^2 x\frac{dy}{dx} - (\tan 2x)\,y = \cos^4 x$

$R = e^{\int -\frac{2\tan x}{1-\tan^2 x}\sec^2 x\,dx} = (1 - \tan^2 x)$

After solving differential equation

$2y(1 - \tan^2 x) = \sin(2x) + c$

$\Rightarrow c = 0 \;\Rightarrow y = \frac{1}{2}\tan 2x.\cos^2 x$

$\Rightarrow y\left(\frac{\pi}{6}\right) = \frac{1}{2}.\sqrt{3}.\frac{3}{4} = \frac{3\sqrt{3}}{8}$

$\Rightarrow \frac{64y\left(\frac{\pi}{6}\right)}{3\sqrt{3}} = 8$

Q.9 $\alpha + \beta = [1]$

$y' + P(x).y = Q(x)$

y_1 & y_2 are two solution of above equation so

$y_1' + P(x)\,y_1 = Q(x)$ ...(1)

$y_2' + P(x)\,y_2 = Q(x)$ ...(2)

multiply equation (1) by α and equation (2) by β then add

$(\alpha y_1' + \beta y_2') + P(x)(\alpha y_1 + \beta y_2) = Q(x)(\alpha + \beta)$

let $y = \alpha y_1 + \beta y_2$

$y' + P(x)y = Q(x)(\alpha + \beta)$

for y to be solution of diff. equation

$\alpha + \beta = 1$

Q.10 [64]

$y - x = m(X - x)$

$X_{int} = x - \frac{y}{m}$

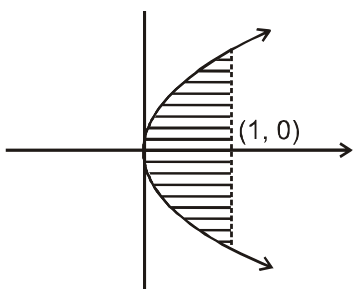
$\frac{x - \frac{y}{m} + x}{2} = 0$

$2x = \frac{y}{m}$

$\frac{dy}{dx} = \frac{y}{2x}$

$\ell n\, y = \frac{1}{2}\ell nx + c \Rightarrow \frac{y^2}{x} = k \Rightarrow y^2 = kx$

curve passes through (1, 2)



$\Rightarrow \quad k = 4$

$\Rightarrow \quad y^2 = 4x$

$A = 2\int_0^1 2\sqrt{x}\, dx$

$A = \frac{8}{3} \quad \Rightarrow \quad 9A^2 = 64$

KVPY
PREVIOUS YEAR'S

Q.1 (D)

$\ell n(x^2 - 1)$

Define for $(x^2 - 1) > 0$

$S:\ X \in (-\infty, -1) \cup (1, \infty)$

$f'(x) = \ell n(x^2 - 1)$

$\int f'(dx) = \int \ell n(x^2 - 1)\, dx$

$f'(x) = \ell n(x^2 - 1).\ x - \int \frac{2x.x}{x^2 - 1} dx$

$= x\,\ell n(x^2 - 1) - \int \frac{2x^2 - 2 + 2}{x^2 - 1} dx$

$= x\,\ell n(x^2 - 1) - 2x - 2x\ \frac{1}{2}\ell n\left(\frac{x-1}{x+1}\right) + C$

$f(x) = x\,\ell n(x^2 - 1) - 2x - \ell n\left(\frac{x-1}{x+1}\right) + C$

$f(2) = x\,\ell n 3 - 4 - \ell n\left(\frac{1}{3}\right) + C = \Rightarrow 2\,\ell n 3 - 4 + \ell n 3 +$

$C = 0$

$C = 4 - 3 \text{ In } 3$

$f(x) = x\text{In}(x^2 - 1) - 2x - \text{In}\left(\frac{x-1}{x+1}\right) + 4 - 3\text{ In } 3$ defined for S

Infinite C values posible in set S such that $f'(x) = \text{In}(x^2 - 1)$

Q.2 (D)

$\int_0^x f(t)dt\ \phi(f(x))$ .....(1)

Differentiable both side with respect to x

$f(x) = pf'(x)$

$\frac{f'(x)}{f(x)} = \frac{1}{p}$

Now, integrating both side with w.r.t x.

$\text{Inf}(x) = \frac{x}{p} + C$

$f(x) = k.e^{x/p}$ ....(2)

patting x = 0 in original equation (1)

$0 = pf(0)$

$\Rightarrow$ either $p = 0$ or $\{f(0) = 0\ \&\ p \neq 0\}$

Let case ; I where $f(0) = 0\ \&\ \phi \neq 0$

$\Rightarrow k = 0$ then from equation (2)

$f(x) = 0 \qquad \Rightarrow f(x) = 0$

$\Rightarrow$ i.e. $p \neq 0$ then there is no non zero continuous to f (x)

case. II p = 0

$\int_0^x f(t)dt\ 0 \forall x \in R$

It is only possible when $f(x) = 0$

Thence $\forall p \in R$: There is no nonzero continuous function. satisfying the given condition.

Hence $S \in R$

Q.3 (D)

$(f'(x))^2 = \pm 4f(x)$

Let f(x) is non negative in its domain

$\Rightarrow (f'(x))^2 = 4f(x) \qquad \Rightarrow f'(x) = \pm 2\sqrt{f(x)}$

$\Rightarrow \int \frac{f'(x)}{\sqrt{f(x)}} dx = \int \pm 2dx \quad (f(x) \neq 0)$

$\Rightarrow 2\sqrt{f(x)} = \pm 2x \qquad f(0) = 0)$

$\Rightarrow \sqrt{f(x)} = \pm x \Rightarrow f(x) = x^2$

so f(x) can be 0 or x^2
proceeding in same way, we can also have $f(x) = -x^2$

Also $f(x) = \begin{cases} x^2, & x < 0 \\ -x^2, & x \ge 0 \end{cases}$ or $f(x) = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$ etc

JEE-MAIN PREVIOUS YEAR'S

Q.1 (2)

$$\frac{dy}{dx} = \frac{(x-2)^2 + y + 4}{(x-2)} = (x-2) + \frac{y+4}{(x-2)}$$

Let $x - 2 = t \Rightarrow dx = dt$
and $y + 4 = u \Rightarrow dy = du$

$$\frac{dy}{dx} = \frac{du}{dt}$$

$$\frac{du}{dt} = t + \frac{u}{t} \Rightarrow \frac{du}{dt} - \frac{u}{t} = t$$

$$\text{I.F} = \int^{\frac{-1}{t}dt} = e^{-\ln t} = \frac{1}{t}$$

$$u.\frac{1}{t} = \int t.\frac{1}{t}\, t\, dt \Rightarrow \frac{u}{t} = t + c$$

$$\frac{y+4}{x-2} = (x-2) + c$$

Passing through (0, 0)
$c = 0$
$\Rightarrow (y + 4) = (x - 2)^2$

Q.2 [0.5]

$(2xy^2 - y)\, dx = -x\, dy$

$$x\frac{dy}{dx} = y - 2xy^2$$

$$\frac{dy}{dx} = \frac{1}{x}.y - 2xy^2$$

$$\frac{dy}{dx} - \frac{1}{x}.y = -2xy^2$$

$$y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = -2$$

$$y^{-1} = t \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$-\frac{dt}{dx} - \frac{1}{x}t = -2$$

$$\Rightarrow \frac{dt}{dx} + \frac{1}{x}.t = 2 \qquad \text{I.F.} = e^{\ell nx} = x$$

$$tx = 2 \times \frac{x^2}{2} + c \qquad \Rightarrow \frac{x}{y} = x^2 + c$$

It passes through P(2, 2)
$\therefore \quad c = -3$

$$\therefore \quad \frac{x}{y} = x^2 - 3$$

$$\text{If } x = 1,\ \frac{1}{y} = -2 \qquad \Rightarrow y = -\frac{1}{2}$$

$$\therefore \quad |y(1)| = \frac{1}{2} = 0.5$$

Q.3 (1)

$$\frac{dy}{dx} + \frac{y}{x} = 6x^3$$

$$\text{I.F.} = e^{\int\frac{dx}{x}} = x$$

$$\therefore \quad yx = \int bx^4 dx = \frac{bx^5}{5} + C$$

Passes through (1,2), we get

$$2 = \frac{b}{5} + C \qquad .....(i)$$

$$\text{Also, } \int_1^2\left(\frac{bx^5}{5} + \frac{C}{x}\right)dx = \frac{62}{5}$$

$$\Rightarrow \frac{b}{25} \times 32 + c\,\ell n2 - \frac{b}{25} = \frac{62}{5}$$

$\Rightarrow C = 0\ \&\ b = 10$

Q.4 [2]

order of differential equation is 1.
$2yy' = a$

$$\Rightarrow \quad y^2 = 2yy'\left(x + \sqrt{2yy'}\right)$$

$$\Rightarrow \quad y - 2xy' = 2y'.\sqrt{2yy'}$$

$$\Rightarrow \quad (y - 2xy')^2 = 4(y')^2.2yy'$$

$$\Rightarrow \quad \left(y - 2x.\frac{dy}{dx}\right)^2 = 8y.\left(\frac{dy}{dx}\right)^3$$

Degree of Differential equation = 3

Q.5 [4]

$$\frac{dx}{dt} \propto dt$$

$\Rightarrow \frac{dx}{dt} = \lambda x$

$\Rightarrow \int_{1000}^{x} \frac{dx}{x} = \lambda \int_{0}^{t} dt$

$\Rightarrow \ell n \frac{x}{1000} = \lambda t$

at $t = 2, x = 1200$

$\therefore \quad 2\lambda = \ell n \frac{6}{5}$

$\therefore \quad x = 1000.e^{\frac{1}{2}\ell n\frac{6}{5}.t}$

Now $2000 = 1000.e^{\frac{1}{2}\ell n\frac{6}{5}.\frac{k}{\ell n\frac{5}{6}}}$

$\Rightarrow \quad 2 = e^{\frac{k}{2}}$

$\Rightarrow \quad \frac{k}{2} = -\ell n2$

$\Rightarrow \quad \frac{k}{\ell n2} = -2$

Q.6 (2)

$\frac{dy}{dx} = \frac{xy^2 + y}{x}$

$\Rightarrow \frac{xdy - ydx}{y^2} = x\,dx$

$\Rightarrow -d\left(\frac{x}{y}\right) = d\left(\frac{x^2}{2}\right)$

$\Rightarrow \frac{-x}{y} = \frac{x^2}{2} + C$

It passes through (–2,3)

$\Rightarrow \frac{2}{3} = 2 + C$

$C = \frac{-4}{3}$

$\therefore$ curve is $\frac{-x}{y} = \frac{x^2}{2} - \frac{4}{3}$

It also passes through (3, y)

$\frac{-3}{y} = \frac{9}{2} - \frac{4}{3}$

$\Rightarrow \frac{-3}{y} = \frac{19}{6}$

$\Rightarrow \; y = -\frac{18}{19}$

Q.7 (3)

$\frac{dp(t)}{dt} = \frac{P(t) - 900}{2}$

$\int_0^t \frac{dP(t)}{P(t) - 900} = \int_0^t \frac{dt}{2}$

$\{\ell n \mid P(t) - 900 \mid\}_0^t = \left\{\frac{t}{2}\right\}_0^t$

$\ell n|P(t) - 900| - \ell n|P(0) - 900| = \frac{t}{2}$

$\ell n|P(t) - 900| - \ell n50 = \frac{t}{2}$

Let at $t = t_1$, $P(t) = 0$ hence

$\ell n|P(t) - 900| - \ell n\,50 = \frac{t_1}{2}$

$t_1 = 2\,\ell n18$

Q.8 (4)

$\frac{dy}{dx} + 2y\tan x = \sin x$

I.F. $= e^{\int 2\tan x dx} = e^{2\ell n \sec x}$

I.F. $= \sec^2 x$

$y.(\sec^2 x) = \int \sin x.\sec^2 x dx$

$y.(\sec^2 x) = \int \sec x \tan x dx$

$y.(\sec^2 x) = \sec x + C$

$x = \frac{\pi}{3};\ y = 0$

$\Rightarrow \quad C = -2$

$\Rightarrow \quad y = \frac{\sec x - 2}{\sec^2 x} = \cos x - 2\cos^2 x$

$y = t - 2t^2 \Rightarrow \frac{dy}{dt} = 1 - t = 0 \Rightarrow t = \frac{1}{4}$

$\therefore \max = \frac{1}{4} - \frac{1}{8} = \frac{2-1}{8} = \frac{1}{8}$

Q.9 (2)

$\frac{dy}{dx} + (\tan x)y = \sin x;\ 0 \le x \le \frac{\pi}{3}$

I.F. $= e^{\int \tan x\, dx} = e^{\ell n \sec x} = \sec x$

$y\sec x = \int \tan x\, dx$

$y\sec x = \int \tan x dx$

$y\sec x = \ln|\sec x| + C$

$x = 0, y = 0 \Rightarrow \therefore c = 0$

$y\sec x = \ln|\sec x|$

$y = \cos x \cdot \ln|\sec x|$

$$y\Big|_{x=\frac{\pi}{4}} = \left(\frac{1}{\sqrt{2}}\right)\cdot \ell n\sqrt{2}$$

$$y\Big|_{x=\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}\log_e 2$$

Q.10 (2)

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}, \; x \in (0, \infty)$$

put $y = vx$

$$x\frac{dv}{dx} + v = \frac{v^2 - 1}{2v}$$

$$\frac{2v}{v^2+1}dv = -\frac{dx}{x}$$

Integrate,

$ln(v^2 + 1) = - lnx + C$

$$ln\left(\frac{y^2}{x^2}+1\right) = -lnx + C$$

put $x = 1, y = 1, C = ln2$

$$ln\left(\frac{y^2}{x^2}+1\right) = -lnx + ln2$$

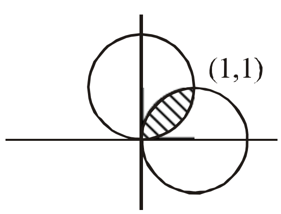
$\Rightarrow x^2 + y^2 - 2x = 0$ (Curve C_1)

Similarly,

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

Put $y = vx$

$x^2 + y^2 - 2y = 0$



$$\text{required area} = 2\int_0^1\left(\sqrt{2x - x^2} - x\right)dx = \frac{\pi}{2} - 1$$

Q.11 (1)

$$\frac{dy}{dx} = (1+y)(x-1)$$

$$\frac{dy}{(y+1)} = (x-1)dx$$

$$\text{Integrate } \ln(y+1) = \frac{x^2}{2} - x + c$$

$$(0,0) \Rightarrow c = 0 \Rightarrow y = e^{\left(\frac{x^2}{2}-x\right)} - 1$$

Q.12 (2)

$\cos x(3\sin x + \cos x + 3)dy$

$= (1 + y\sin x(3\sin x + \cos x + 3))dx$

$$\frac{dy}{dx} - (\tan x)y = \frac{1}{(3\sin x + \cos x + 3)\cos x}$$

$$\text{I.F.} = e^{\int -\tan x\, dx} = e^{\ell n|\cos x|} = |\cos x|$$

$$= \cos x \;\forall\; x \in \left[0, \frac{\pi}{2}\right)$$

Solution of D.E.

$$y(\cos x) = \int(\cos x)\cdot\frac{1}{\cos x(3\sin x + \cos x + 3)}dx + C$$

$$y(\cos x) = \int\frac{dx}{3\sin x + \cos x + 3}dx + C$$

$$y(\cos x) = \int\frac{\left(\sec^2\frac{x}{2}\right)}{2\tan^2\frac{x}{2} + 6\tan\frac{x}{2} + 4}dx + C$$

Now

$$\text{Let } I_1 = \int\frac{\left(\sec^2\frac{x}{2}\right)}{2\left(\tan^2\frac{x}{2} + 3\tan\frac{x}{2} + 2\right)}dx + C$$

$$\text{Put } \tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

$$I_1 = \int\frac{dt}{t^3 + 3t + 2} = \int\frac{dt}{(t+2)(t+1)}$$

$$= \int\left(\frac{1}{t+1} - \frac{1}{t+2}\right)dt$$

$$= \ell n\left|\left(\frac{t+1}{t+2}\right)\right| = \ell n\left|\left(\frac{\tan\frac{x}{2}+1}{\tan\frac{x}{2}+2}\right)\right|$$

So solution of D.E.

$$y(\cos x) = \ell n\left|\frac{1+\tan\frac{x}{2}}{2+\tan\frac{x}{2}}\right| + C$$

$$\Rightarrow\ y(\cos x)=\ell n\left(\frac{1+\tan\frac{x}{2}}{2+\tan\frac{x}{2}}\right)+C \quad \text{for } 0\le x<\frac{\pi}{2}$$

Now, it is given y(0) = 0

$$\Rightarrow\ 0=\ell n\left(\frac{1}{2}\right)+C \qquad \Rightarrow \boxed{C=\ell n 2}$$

$$\Rightarrow\ y(\cos x)=\ell n\left(\frac{1+\tan\frac{x}{2}}{2+\tan\frac{x}{2}}\right)+\ell n 2$$

For $x=\frac{\pi}{3}$

$$y\left(\frac{1}{2}\right)=\ell n\left(\frac{1+\frac{1}{\sqrt{3}}}{2+\frac{1}{\sqrt{3}}}\right)+\ell n 2$$

$$y=2\ell n\left(\frac{2\sqrt{3}+10}{11}\right)$$ **Ans.(2)**

Q.13 (3)

$$\frac{dy}{dx}-\frac{y}{2x}=\frac{x^{9/4}}{x^{5/4}(x^{3/4}+1)}$$

$$IF=e^{-\int\frac{dx}{2d}}=e^{-\frac{1}{2}\ln x}=\frac{1}{x^{1/2}}$$

$$y.x^{-1/2}=\int\frac{x^{9/4}\cdot x^{-1/2}}{x^{5/4}(x^{3/4}+1)}dx$$

$$\int\frac{x^{1/2}}{(x^{3/4}+1)}dx$$

$x = t^4 \Rightarrow dx = 4t^3\,dt$

$$\int\frac{t^2\cdot 4t^3\,dt}{(t^3+1)}$$

$$4\int\frac{t^2(t^3+1-1)}{(t^3+1)}dt$$

$$4\int t^2dt-4\int\frac{t^2}{t^3+1}dt$$

$$\frac{4t^3}{3}-\frac{4}{3}\ln(t^3+1)+C$$

$$yx^{-1/2}=\frac{4x^{3/4}}{3}-\frac{4}{3}\ln(x^{3/4}+1)+C$$

$$1-\frac{4}{3}\log_e 2=\frac{4}{3}-\frac{4}{3}\log_e 2+C$$

$$\Rightarrow C=-\frac{1}{3}$$

$$y=\frac{4}{3}x^{5/4}-\frac{4}{3}\sqrt{x}\ln(x^{3/4}+1)-\frac{\sqrt{x}}{3}$$

$$y(16)=\frac{4}{3}\times 32-\frac{4}{3}\times 4\ln 9-\frac{4}{3}$$

$$=\frac{124}{3}-\frac{32}{3}\ln 3=4\left(\frac{31}{3}-\frac{8}{3}\ln 3\right)$$

Q.14 (3)

y2 = 4ax + 4a2

differentiate with respect to x

$$\Rightarrow 2y\frac{dy}{dx}\quad 4a$$

$$\Rightarrow a=\left(\frac{y}{2}\frac{dy}{dx}\right)$$

so, required differential equation is

$$y^2=\left(4\times\frac{y}{2}\frac{dy}{dx}\right)x+4\left(\frac{y}{2}\frac{dy}{dx}\right)^2$$

$$\Rightarrow y^2\left(\frac{dy}{dx}\right)^2+2xy\left(\frac{dy}{dx}\right)-y^2=0$$

$$\Rightarrow y\left(\frac{dy}{dx}\right)^2+2x\left(\frac{dy}{dx}\right)-y=0$$

Q.15 (1)

Let y + 1 = Y

$$\therefore\ \frac{dY}{dx}=Y^2e^{\frac{x^2}{2}}-xY$$

Put $-\frac{1}{Y}=k$

$$\Rightarrow\frac{dk}{dx}+k(-x)=e^{\frac{x^2}{2}}$$

I.F. = $e^{-\frac{x^2}{2}}$

$$\therefore\ k=(x+c)\,e^{x^2/2}$$

Put $k=-\frac{1}{y+1}$

$$\therefore y+1=-\frac{1}{(x+c)e^{x^2/2}} \qquad ...(i)$$

when x = 2, y = 0, then c = –2 – $\frac{1}{e^2}$

diffentiate equation (i) & put x = 1

We get $\left(\frac{dy}{dx}\right)_{x=1}=-\frac{e^{3/2}}{(1+e^2)^2}$

Q.16 (3)

Q.17 [4]

Q.18 (4)

Q.19 [4]

Q.20 (1)

Q.21 (1)

Q.22 [2]

Q.23 [2]

Q.24 [2]

Q.25 (1)

Q.26 (2)

Q.27 [2]

Q.28 (4)

Q.29 (1)

Q.30 (2)

Q.31 (2)

Q32 (3)

Q.33 (4)

Q.34 (4)

Q.35 (3)

Q.36 (1)

Q.37 (1)

JEE-ADVANCED PREVIOUS YEAR'S

Q.1 (Bonus)

Data inconsistent.

Putting x = 1 , in given integral equation $\Rightarrow f(1) = 1/3$, a contradiction (given that f(1) = 2).

However if considering integral equation as

$$6\int_1^x f(t)\,dt = 3xf(x) - x^3 - 5$$

we obtain correct answer.

Differentiating the integral equation

$6f(x) = 3f(x) + 3xf'(x) - 3x^2$

$f'(x) - \frac{1}{x} f(x) = x$

put y = f(x)

$\frac{dy}{dx} - \frac{1}{x} y = x$

$I.F. = \frac{1}{x}$

General solution is $y\frac{1}{x} = x + c$

Put x = 1, y = 2 $\Rightarrow$ c = 1

$\Rightarrow \quad y = x^2 + x$

$f(x) = x^2 + x$

$f(2) = 4 + 2 = 6$

Q.2 [0]

$y'(x) + y(x)\,g'(x) = g(x)\,g'(x),\ y(0) = 0\ \forall x \in R$

$\frac{d}{dx}(y(x)) + y(x)\,g'(x) = g(x)\,g'(x).,$

$g(0) = g(2) = 0.$

$I{\cdot}F = e^{\int g'(x)dx} = e^{g(x)}$

$y(x)\,e^{g(x)} = \int e^{g(x)} g(x) g'(x)\,dx + c$

Let g(x) = t

$g'(x)\,dx = dt$

$y(x)\,e^{g(x)} = \int te^t\,dt$

$= te^t - e^t + c$

$y(x) = (g(x)-1) + c\,e^{-g(x)}$

Let x = 0 $y(0) = (g(0)-1) + c\,e^{-g(0)}$

$0 = (0-1) + c \quad \Rightarrow c = 1$

$y(x) = (g(x)-1) + e^{-g(x)}$

$y(2) = (g(2)-1) + e^{-g(2)}$

$y(2) = (0-1) + e^{-(0)} = -1 + 1 = 0$

Q.3 (A)

$\frac{dy}{dx} - y\tan x = 2x\sec x$

$y(0) = 0$

$I.F. = e^{-\int \tan x\,dx} = e^{-\log\sec x}$

I.F. = cos x

$\cos x \,.\, y = \int 2x \sec x.\cos x\,dx$

$\cos x.\ y = x^2 + c$

c = 0

$y = x^2 \sec x$

$y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16}.\sqrt{2} = \frac{\pi^2}{8\sqrt{2}}$

$y'\left(\frac{\pi}{4}\right) = \frac{\pi}{2}.\sqrt{2} + \frac{\pi^2}{16}.\sqrt{2}$

$y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}.2 = \frac{2\pi^2}{9}$

$y'\left(\frac{\pi}{3}\right) = 2\frac{\pi}{2}.2 + \frac{\pi^2}{9}.2.\sqrt{3}$

$\frac{4\pi}{3} + \frac{2\pi^2\sqrt{3}}{9}$

Q.4 (A)

Given slope at (x, y) is

$\frac{dy}{dx} = \frac{y}{x} + \sec(y/x)$

let $\frac{y}{x} = t \Rightarrow y = xt \quad \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$t + x\frac{dt}{dx} = t + \sec(t)$

$\int \cos t\, dt = \int \frac{1}{x}dx$

$\sin t = \ell n\, x + c$

$\sin(y/x) = \ell n\, x + c$

This curve passes through $(1, \pi/6)$

$\sin(\pi/6) = \ell n(1) + c \Rightarrow c = 1/2$

$\sin(y/x) = \ell n\, x + 1/2$

Q.5 (B)

I.F. $= e^{\int \frac{x}{x^2-1}dx} = e^{\frac{1}{2}\int \frac{2x}{x^2-1}dx} = e^{\frac{1}{2}\ell n|x^2-1|} =$

$e^{\frac{1}{2}\ell n(1-x^2)} = \sqrt{1-x^2}$

$\therefore \quad y\sqrt{1-x^2} = \int \frac{x^4+2x}{\sqrt{1-x^2}} \times \sqrt{1^2-x^2}\,dx + c$

$y\sqrt{1^2-x^2} = \frac{x^5}{5} + x^2 + c$

$x = 0,\ y = 0 \Rightarrow c = 0$

$y = \frac{\frac{x^5}{5}+x^2}{\sqrt{1-x^2}}$

$\therefore \quad I = \int_0^{\frac{\sqrt{3}}{2}} \left(\frac{\frac{x^5}{5}+x^2}{\sqrt{1-x^2}} + \frac{\frac{-x^5}{5}+x^2}{\sqrt{1-x^2}} \right) dx$

$= 2\int_0^{\frac{\sqrt{3}}{2}} \frac{x^2}{\sqrt{1-x^2}}dx$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{3}} \frac{\sin^2\theta\cos\theta}{\cos\theta}d\theta$

$= \int_0^{\frac{\pi}{3}} (1-\cos 2\theta)\,d\theta = \left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{3}} =$

$\frac{\pi}{3} - \frac{1}{2}\times\sin\frac{2\pi}{3} = \frac{\pi}{3} - \frac{1}{2}\times\frac{\sqrt{3}}{2} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

Q.6 (B)

$f'(x) = 2x\,f(x)$

$\frac{f'(x)}{f(x)} = 2x$

$\ell n(f(x)) = x^2 + c$

$x = 0,\ f(0) = 1$

$c = 0$

$\therefore \quad \ell n(f(x)) = x^2$

$f(x) = e^{x^2}$

$\therefore \quad F(x) = f(x) + c$

$F(x) = e^{x^2} + c$

$F(0) = 0$

$\therefore \quad c = -1$

$\therefore \quad f(x) = e^{x^2} - 1$

$f(2) = e^4 - 1.$

Q.7 (A,C)

$(1 + e^x)\frac{dy}{dx} + ye^x = 1$

$\frac{dy}{dx} + \frac{e^x}{1+e^x}y = \frac{1}{1+e^x}$

I.F $= e^{\int \frac{e^x}{1+e^x}dx} = e^{\ell n(1+e^x)} = 1 + e^x$

complete solution

$y.(1 + e^x) = \int 1dx$

$(1 + e^x)y = x + c$

$x = 0,\ y = 2 \Rightarrow c = 4$

$(1 + e^x)y = x + 4$

$y = \frac{x+4}{e^x+1}$

$x = -4,\ y = 0$

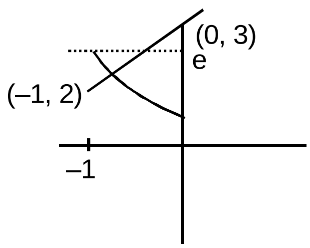
$x = -2, y = \frac{2}{x^{-2}+1}$

$\frac{dy}{dx} = \frac{(e^x+1).1-(x+4)e^x}{(e^x+1)^2}$

$\frac{e^x(-x-3)+1}{(e^x+1)^2}$

$\frac{dy}{dx} = 0$

$\Rightarrow \quad x + 3 = e^{-x}$



$e^x = \frac{1}{x+3}$

Q.8 (B,C)

$(x-\alpha)^2 + (y-\alpha)^2 = r^2$

$x^2 + y^2 - 2\alpha x - 2\alpha y + 2\alpha^2 - r^2 = 0$

$2x + 2yy' - 2\alpha - 2\alpha y' = 0$ ...(i)

$\Rightarrow \quad \alpha = \frac{x+yy'}{1+y'}$ ...(ii)

again diff. w.r.t

$2 + 2(y')2 + 2yy'' - 2\alpha y'' = 0$

$\Rightarrow \quad 1 + (y')^2 + yy' - \left(\frac{x+yy'}{1+y'}\right)y'' = 0$

$\Rightarrow \quad 1 + y' + (y')^2 + (y')^3 + yy'' + yy'y' - xy'' - yy'y'' = 0$

$\Rightarrow \quad (y-x)y'' + (1 + y + (y')^2)\, y' + 1 = 0$

$\Rightarrow \quad P = y - x,\ Q = 1 + y' + (y')^2$

Ans. (B,C)

Note : P & Q will not be unique function as

$Py'' + Qy' + Ry' - Ry' + 1 = 0$

$\Rightarrow \quad \frac{Py'}{1-Ry''} + \frac{Qy'}{1-Ry'} + 1 = 0$ Hence new P & Q can be obtained.

So it can be a controversial problem.

$Py'' + Qy' + Ry' - Ry' + 1 = 0$

$\Rightarrow \quad \frac{Py'}{1-Ry''} + \frac{Qy'}{1-Ry'} + 1 = 0$

Q.9 (A)

$f'(x) = 2 - \frac{f(x)}{x}$

$\Rightarrow \frac{dy}{dx} = 2 - \frac{y}{x}$

$\Rightarrow y = x + \frac{C}{x}$

$\Rightarrow f(1) \neq 1$

$\Rightarrow C \neq 0$

$f(x) = x + \frac{C}{x}$

$f'(x) = 1 - \frac{C}{x^2}$

$f'\left(\frac{1}{x}\right) = 1 - Cx^2$

Hence, $\lim_{x\to 0+} f'\left(\frac{1}{x}\right) = 1$

Q.10 (A, D)

$\left(x^2 + xy + 4x + 2y + 4\right)\frac{dy}{dx} - y^2 = 0$

$\left[(x+2)^2 + y(x+2)\right]\frac{dy}{dx} = y^2$

$\frac{dy}{dx} = \frac{y^2}{(x+2)(x+2+y)}$

Let, $y = (x+2)v$

$\frac{dy}{dx} = v + (x+2)\frac{dv}{dx}$

$\ln|v| + v = -\ln|x+2| + C$

$\ln\left|\frac{y}{x+2}\right| + \frac{y}{x+2} = -\ln|x+2| + C$

$\downarrow (1,3)$

$\Rightarrow C = 1 + \ln 3$

$\therefore \ln|y| + \frac{y}{x+2} = 1 + \ln 3$

Hence, (a, d)

Q.11 (B)

$\frac{dy}{dx} = \frac{\left(\sqrt{4+\sqrt{9+x}}\right)^{-1}}{8\sqrt{x}\sqrt{9+\sqrt{x}}}$

$dy = \frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}} \cdot \frac{1}{\sqrt{9}+\sqrt{x}} \cdot \frac{1}{8\sqrt{x}}dx$

Let $4+\sqrt{9+\sqrt{x}}=t \Rightarrow \frac{1}{2\sqrt{9+\sqrt{x}}}\times\frac{1}{2\sqrt{x}}dx=dt$

$\int dy=\int\frac{1}{\sqrt{t}}\cdot\frac{1}{2}dt$

$y=\sqrt{t}+c$

$y=\sqrt{4+\sqrt{9+\sqrt{x}}}+c$

at $x=0$, $y=\sqrt{7}$

$\Rightarrow \sqrt{7}=\sqrt{7}+c \Rightarrow c=0$

$y=\sqrt{4+\sqrt{9+\sqrt{x}}}$

at $x=256 \Rightarrow y=\sqrt{4+\sqrt{9+\sqrt{256}}}=3$

Q.12 [0.4]

$\frac{dy}{dx}=25y^2-4$

So, $\frac{dy}{25y^2-4}=dx$

Integrating, $\frac{1}{25}\times\frac{1}{2\times\frac{2}{5}}\ln\left|\frac{y-\frac{2}{5}}{y+\frac{2}{5}}\right|=x+c$

$\Rightarrow \ln\left|\frac{5y-2}{5y+2}\right|=20(x+c)$

Now, c = 0 as f(0) = 0

Hence $\left|\frac{5y-2}{5y+2}\right|=e^{(20x)}$

$\underset{x\to-\infty}{\text{let}}\left|\frac{5f(x)-2}{5f(x)+2}\right|=\underset{x\to-\infty}{\text{let}}\ e^{(20x)}$

Now, RHS = 0 $\Rightarrow \underset{x\to-\infty}{\text{let}}(5f(x)-2)=0$

$\Rightarrow \underset{x\to-\infty}{\text{let}}\ f(x)=\frac{2}{5}$

Q.13 (A,C)

Integrating factor = $e^{\alpha x}$

So $ye^{\alpha x}=\int xe^{(\alpha+\beta)x}dx$

Case - I

If $\alpha+\beta=0$ $\qquad ye^{\alpha x}=\frac{x^2}{2}+c$

It passes through (1, 1) $\Rightarrow C=e^{\alpha}-\frac{1}{2}$

So $ye^{\alpha x}=\frac{x^2-1}{2}+e^{\alpha}$

For $\alpha=1$

$y=\frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x}\rightarrow(A)$

Case - II

If $\alpha+\beta\neq0$

$ye^{\alpha x}=\frac{x.e^{(\alpha+\beta)x}}{\alpha+\beta}-\frac{1}{\alpha+\beta}e^{(\alpha+\beta)x}dx$

$\Rightarrow ye^{\alpha x}=\frac{xe^{(\alpha+\beta)x}}{\alpha+\beta}-\frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2}+c$

So $c=e^{\alpha}-\frac{e^{\alpha+\beta}}{\alpha+\beta}+\frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}$

$y=\frac{e^{\beta x}}{(\alpha+\beta)^2}\left((\alpha+\beta)x-1+e^{-\alpha x}\right)\left(e^{x}-\frac{e^{\alpha+\beta}}{\alpha+\beta}+\frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}\right)$

If $\alpha=\beta=1$

$y=\frac{e^x}{4}(2x-1)+e^{-x}\left(e-\frac{e^2}{2}+\frac{e^2}{4}\right)$

$y=\frac{e^x}{2}\left(x-\frac{1}{2}\right)+e^{-x}\left(e-\frac{e^2}{4}\right)\rightarrow(c)$